

Resultants over Commutative Idempotent Semirings

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Abstract

The resultant plays a crucial role in (computational) algebra and algebraic geometry. One of the most important and well known properties of the resultant is that it is equal to the determinant of the Sylvester matrix. In 2008, Odagiri proved that a similar property holds over the tropical semiring if one replaces subtraction with addition. The tropical semiring belongs to a large family of algebraic structures called commutative idempotent semiring. In this paper, we prove that the same property (with subtraction replaced with addition) holds over an *arbitrary* commutative idempotent semiring.

1 Introduction

The main contribution of this paper is adapting a certain important property of resultant (over commutative rings) to commutative idempotent semirings. The work was inspired by Odagiri's work [28] where the property of resultant is adapted to a particular commutative idempotent semiring, namely tropical semiring. Below we elaborate on the above statements.

The resultant plays a crucial role in (computational) algebra and algebraic geometry [36, 32, 24, 9]. Let

$$\begin{aligned} f &= (x - \alpha_1) \cdots (x - \alpha_m) \\ g &= (x - \beta_1) \cdots (x - \beta_n) \end{aligned}$$

be two polynomials over a commutative ring. The resultant \mathbf{R} of f and g is defined as

$$\mathbf{R} = \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} (\alpha_i - \beta_j)$$

and the Sylvester expression of f and g is defined as

$$\mathbf{S} = \det M$$

where M is a certain matrix whose entries are from the coefficients of the two polynomials. One of the most important and well known properties of the resultant [36, 10] is that

$$\mathbf{R} = \mathbf{S}.$$

The tropical semiring is a variant of a commutative ring where it is equipped with a total order and that the addition operation is defined as maximum. As the result, it does not allow subtraction (due to lack of additive inverse; hence the name semiring). It has been intensively investigated due to numerous interesting applications [33, 31, 30, 18, 3, 15, 34, 19, 8, 35, 23, 5, 25].

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There have been several adaptations of the properties of the resultant (over commutative rings) to the tropical semiring [27, 11, 4, 37, 28, 20]. In particular, Odagiri [28] proved that the property of the resultant still holds if one simply replaces subtraction with addition, that is, if we let

$$\begin{aligned} f &= (x + \alpha_1) \cdots (x + \alpha_m) \\ g &= (x + \beta_1) \cdots (x + \beta_n) \end{aligned}$$

and redefine the resultant as

$$\mathbf{R} = \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} (\alpha_i + \beta_j),$$

and redefine the Sylvester expression as

$$\mathbf{S} = \text{per } M$$

then

$$\mathbf{R} = \mathbf{S}.$$

The tropical semiring belongs to a large family of algebraic structures called commutative idempotent semiring (CIS) [29, 21, 16, 13, 12]. As the name indicates, a commutative idempotent semiring is similar to a commutative ring, except that we do not require subtraction (additive inverse) but we instead require additive idempotency. There are many interesting algebraic structures that are CIS (see Section 1).

Hence one naturally wonders whether we can extend Odagiri's result on tropical semiring to the whole family of CIS. The main contribution of this paper is to answer affirmatively, proving that Odagiri's result indeed holds for arbitrary CIS, not just for the tropical semiring.

For proving the property, we, at the beginning, naturally attempted to generalize/relax the proof technique of Odagiri. However we found it practically impossible mainly because Odagiri's proof crucially exploits the fact that the tropical semiring has a total order. Since CIS, in general, does not require a total order, we had to develop a different proof technique. The new technique consists of the following four parts:

1. Represent each term in \mathbf{R} as a certain boolean matrix, which we call a *res*-representation.
2. Represent each term in \mathbf{S} as a certain pair of boolean matrices, which we call a *syl*-representation.
3. Show that if a term has a *res*-representation then it has a *syl*-representation.
4. Show that if a term has a *syl*-representation then it has a *res*-representation.

The representation of terms as boolean matrices are not essential from logical point of view, but they are extremely helpful in discovering, explaining and understanding the steps of the proof. The proof is constructive, that is, it provides an algorithm that takes a *res*-representation and produces a *syl*-representation, and vice versa. The implementation of the algorithms in Maple [26, 2] can be downloaded from

<http://www.math.ncsu.edu/~hong/rcis/>

Note that all the above discussion is approached and stated algebraically in the sense that we consider the equality between two algebraic objects \mathbf{R} and \mathbf{S} . Over the commutative ring case (in particular integral domain), the algebraic equality has an obvious geometric interpretation, namely two polynomials have common root iff $\mathbf{S} = 0$. Over the tropical semiring case, the algebraic equality can also be given a similar geometric interpretation, using a modified notion of root [28, 19, 5, 25]. Thus one wonders whether the algebraic equality over CIS can be also given a similar geometric interpretation. One way for this would be extending the notion of root further. We leave it as a challenge for future work.

The paper is structured as follows. In Section 2, we recall the axiomatic definition of CIS and list several algebraic structures that satisfy the axioms. In Section 3, we give a precise statement of the main result of this paper. In Section 4, we illustrate the main result on the algebraic structures listed in Section 2. In Section 5, we informally sketch the overall structure/underlying ideas of the proof by using a small case. In Section 6, we finally provide a detailed and general proof of the main result.

2 Review of Commutative Idempotent Semiring

In this section, we review the definition of commutative idempotent semiring, and list a few examples. For more details, see [21, 17, 12, 14].

Definition 1 (Commutative Idempotent Semiring). *A Commutative Idempotent Semiring (CIS) is a tuple $(\mathcal{I}, +, \times, 0, 1)$ where \mathcal{I} is a set, $+$ and \times are binary operations over \mathcal{I} and $0, 1$ are elements of \mathcal{I} such that the following properties hold for all $a, b, c \in \mathcal{I}$:*

$+$	\times	$+$ and \times
$a + b = b + a$	$a \times b = b \times a$	$(a + b) \times c = a \times c + b \times c$
$(a + b) + c = a + (b + c)$	$(a \times b) \times c = a \times (b \times c)$	
$a + 0 = a$	$a \times 1 = a$	
$a + a = a$	$a \times 0 = 0$	

Remark 1. *Note that CIS does not require the existence of additive inverse, thus semiring. It instead requires idempotency, thus idempotent semiring.*

Example 1. *We list a few examples of commutative idempotent semirings (CIS).*

name	\mathcal{I}	$+$	\times	0	1
tropical	$\mathbb{R} \cup \{-\infty\}$	maximum	addition	$-\infty$	0
power set	power set of a set S	union	intersection	\emptyset	S
topology	topology on a set S	union	intersection	\emptyset	S
compact-convex	compact convex subsets of \mathbb{R}^n	convex hull	Minkowski sum	\emptyset	$\{0\}$
sequences	sequences over a CIS A	component-wise	convolution	$(0_A, \dots)$	$(1_A, 0_A, \dots)$
ideals	ideals of a comm. ring R with unity	ideal sum	ideal product	$\{0_R\}$	$\langle 1_R \rangle$

For the sequences,

$$\begin{aligned}
 \text{component-wise} & : \quad \forall i \geq 0 \quad (u + v)_i = u_i +_A v_i \\
 \text{convolution} & : \quad \forall i \geq 0 \quad (u \times v)_i = \sum_{A, j, k \geq 0, j+k=i} u_j \times_A v_k
 \end{aligned}$$

3 Main Result

In this section, we give a precise statement of the main result of this paper and, in the next section, we illustrate it on a few examples. Let \mathcal{I} be a commutative idempotent semiring (CIS). Let $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n \in \mathcal{I}$. Let

$$\begin{aligned}
 f &= \prod_{i=1}^m (x + \alpha_i) \in \mathcal{I}[x] \\
 g &= \prod_{j=1}^n (x + \beta_j) \in \mathcal{I}[x]
 \end{aligned}$$

Definition 2 (Resultant). *The resultant \mathbf{R} of f and g is defined as*

$$\mathbf{R} = \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} (\alpha_i + \beta_j) \in \mathcal{I}.$$

Example 2. Let $m = 3$ and $n = 2$. Then

$$\mathbf{R} = (\alpha_1 + \beta_1)(\alpha_1 + \beta_2)(\alpha_2 + \beta_1)(\alpha_2 + \beta_2)(\alpha_3 + \beta_1)(\alpha_3 + \beta_2)$$

Definition 3 (Sylvester Matrix). Let $a_0, \dots, a_m \in \mathcal{I}$ and $b_0, \dots, b_n \in \mathcal{I}$ be such that

$$f = \sum_{i=0}^m a_{m-i} x^i$$

$$g = \sum_{j=0}^n b_{n-j} x^j$$

Then the Sylvester matrix of f and g is defined as

$$M = \underbrace{\left[\begin{array}{cccccc} a_0 & \cdots & \cdots & \cdots & a_m & \\ & \ddots & & & & \ddots \\ & & a_0 & \cdots & \cdots & a_m \\ b_0 & \cdots & \cdots & b_n & & \\ & \ddots & & & \ddots & \\ & & \ddots & & & \ddots \\ & & & b_0 & \cdots & b_n \end{array} \right]}_{m+n} \left. \begin{array}{l} \left. \vphantom{\begin{matrix} a_0 \\ \vdots \\ a_m \end{matrix}} \right\} n \\ \left. \vphantom{\begin{matrix} b_0 \\ \vdots \\ b_n \end{matrix}} \right\} m \end{array} \right\}$$

Example 3. Let $m = 3$ and $n = 2$. Then

$$f = (x + \alpha_1)(x + \alpha_2)(x + \alpha_3)$$

$$= a_0 x^3 + a_1 x^2 + a_2 x^1 + a_3 x^0$$

$$g = (x + \beta_1)(x + \beta_2)$$

$$= b_0 x^2 + b_1 x^1 + b_2 x^0$$

where

$$a_0 = 1$$

$$a_1 = \alpha_1 + \alpha_2 + \alpha_3$$

$$a_2 = \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3$$

$$a_3 = \alpha_1 \alpha_2 \alpha_3$$

$$b_0 = 1$$

$$b_1 = \beta_1 + \beta_2$$

$$b_2 = \beta_1 \beta_2$$

Thus the Sylvester matrix M of f and g is given by

$$M = \begin{bmatrix} 1 & \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3 & \alpha_1 \alpha_2 \alpha_3 & & \\ & 1 & \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 \alpha_2 + \alpha_1 \alpha_3 + \alpha_2 \alpha_3 & \alpha_1 \alpha_2 \alpha_3 & \\ 1 & \beta_1 + \beta_2 & \beta_1 \beta_2 & & & \\ & 1 & \beta_1 + \beta_2 & \beta_1 \beta_2 & & \\ & & 1 & \beta_1 + \beta_2 & \beta_1 \beta_2 & \\ & & & 1 & \beta_1 + \beta_2 & \beta_1 \beta_2 \end{bmatrix}$$

Definition 4 (Sylvester expression). The Sylvester expression \mathbf{S} of f and g is defined as the permanent of the Sylvester matrix of f and g , that is,

$$\mathbf{S} = \text{per}(M) \in \mathcal{I}.$$

Example 4. Let $m = 3$ and $n = 2$. Then

$$\mathbf{S} = \text{per} \begin{bmatrix} 1 & \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 & \alpha_1\alpha_2\alpha_3 & \\ & 1 & \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 & \alpha_1\alpha_2\alpha_3 \\ 1 & \beta_1 + \beta_2 & \beta_1\beta_2 & & \\ & 1 & \beta_1 + \beta_2 & \beta_1\beta_2 & \\ & & 1 & \beta_1 + \beta_2 & \beta_1\beta_2 \end{bmatrix}$$

Theorem 1 (Main Result). $\mathbf{R} = \mathbf{S}$.

Example 5. We illustrate the “meaning” of the main result (Theorem 1) by verifying it on small degrees: $m = 3$ and $n = 2$ where \mathbf{R} and \mathbf{S} are described in Example 2 and 4. When we expand \mathbf{R} and \mathbf{S} , we obtain

$$\begin{aligned} \mathbf{R} = & \alpha_1^2\alpha_2^2\alpha_3^2 + \alpha_1^2\alpha_2^2\alpha_3\beta_1 + \alpha_1^2\alpha_2^2\alpha_3\beta_2 + \alpha_1^2\alpha_2^2\beta_1\beta_2 + \alpha_1^2\alpha_2\alpha_3^2\beta_1 + \alpha_1^2\alpha_2\alpha_3^2\beta_2 + \alpha_1^2\alpha_2\alpha_3\beta_1^2 + \\ & 2\alpha_1^2\alpha_2\alpha_3\beta_1\beta_2 + \alpha_1^2\alpha_2\alpha_3\beta_2^2 + \alpha_1^2\alpha_2\beta_1^2\beta_2 + \alpha_1^2\alpha_2\beta_1\beta_2^2 + \alpha_1^2\alpha_3^2\beta_1\beta_2 + \alpha_1^2\alpha_3\beta_1^2\beta_2 + \alpha_1^2\alpha_3\beta_1\beta_2^2 + \\ & \alpha_1^2\beta_1^2\beta_2^2 + \alpha_1\alpha_2^2\alpha_3^2\beta_1 + \alpha_1\alpha_2^2\alpha_3^2\beta_2 + \alpha_1\alpha_2^2\alpha_3\beta_1^2 + 2\alpha_1\alpha_2^2\alpha_3\beta_1\beta_2 + \alpha_1\alpha_2^2\alpha_3\beta_2^2 + \alpha_1\alpha_2^2\beta_1^2\beta_2 + \alpha_1\alpha_2^2\beta_1\beta_2^2 + \\ & \alpha_1\alpha_2\alpha_3^2\beta_1^2 + 2\alpha_1\alpha_2\alpha_3^2\beta_1\beta_2 + \alpha_1\alpha_2\alpha_3^2\beta_2^2 + \alpha_1\alpha_2\alpha_3\beta_1^3 + 3\alpha_1\alpha_2\alpha_3\beta_1^2\beta_2 + 3\alpha_1\alpha_2\alpha_3\beta_1\beta_2^2 + \alpha_1\alpha_2\alpha_3\beta_2^3 + \\ & \alpha_1\alpha_2\beta_1^3\beta_2 + 2\alpha_1\alpha_2\beta_1^2\beta_2^2 + \alpha_1\alpha_2\beta_1\beta_2^3 + \alpha_1\alpha_3^2\beta_1^2\beta_2 + \alpha_1\alpha_3^2\beta_1\beta_2^2 + \alpha_1\alpha_3\beta_1^3\beta_2 + 2\alpha_1\alpha_3\beta_1^2\beta_2^2 + \alpha_1\alpha_3\beta_1\beta_2^3 + \\ & \alpha_1\beta_1^3\beta_2^2 + \alpha_1\beta_1^2\beta_2^3 + \alpha_2^2\alpha_3^2\beta_1\beta_2 + \alpha_2^2\alpha_3\beta_1^2\beta_2 + \alpha_2^2\alpha_3\beta_1\beta_2^2 + \alpha_2^2\beta_1^2\beta_2^2 + \alpha_2\alpha_3^2\beta_1^2\beta_2 + \alpha_2\alpha_3^2\beta_1\beta_2^2 + \\ & \alpha_2\alpha_3\beta_1^3\beta_2 + 2\alpha_2\alpha_3\beta_1^2\beta_2^2 + \alpha_2\alpha_3\beta_1\beta_2^3 + \alpha_2\beta_1^3\beta_2^2 + \alpha_2\beta_1^2\beta_2^3 + \alpha_3^2\beta_1^2\beta_2^2 + \alpha_3\beta_1^3\beta_2^2 + \alpha_3\beta_1^2\beta_2^3 + \beta_1^3\beta_2^3 \end{aligned}$$

$$\begin{aligned} \mathbf{S} = & \alpha_1^2\alpha_2^2\alpha_3^2 + \alpha_1^2\alpha_2^2\alpha_3\beta_1 + \alpha_1^2\alpha_2^2\alpha_3\beta_2 + \alpha_1^2\alpha_2^2\beta_1\beta_2 + \alpha_1^2\alpha_2\alpha_3^2\beta_1 + \alpha_1^2\alpha_2\alpha_3^2\beta_2 + \alpha_1^2\alpha_2\alpha_3\beta_1^2 + \\ & 6\alpha_1^2\alpha_2\alpha_3\beta_1\beta_2 + \alpha_1^2\alpha_2\alpha_3\beta_2^2 + \alpha_1^2\alpha_2\beta_1^2\beta_2 + \alpha_1^2\alpha_2\beta_1\beta_2^2 + \alpha_1^2\alpha_3^2\beta_1\beta_2 + \alpha_1^2\alpha_3\beta_1^2\beta_2 + \alpha_1^2\alpha_3\beta_1\beta_2^2 + \\ & \alpha_1^2\beta_1^2\beta_2^2 + \alpha_1\alpha_2^2\alpha_3^2\beta_1 + \alpha_1\alpha_2^2\alpha_3^2\beta_2 + \alpha_1\alpha_2^2\alpha_3\beta_1^2 + 6\alpha_1\alpha_2^2\alpha_3\beta_1\beta_2 + \alpha_1\alpha_2^2\alpha_3\beta_2^2 + \alpha_1\alpha_2^2\beta_1^2\beta_2 + \alpha_1\alpha_2^2\beta_1\beta_2^2 + \\ & \alpha_1\alpha_2\alpha_3^2\beta_1^2 + 6\alpha_1\alpha_2\alpha_3^2\beta_1\beta_2 + \alpha_1\alpha_2\alpha_3^2\beta_2^2 + \alpha_1\alpha_2\alpha_3\beta_1^3 + 9\alpha_1\alpha_2\alpha_3\beta_1^2\beta_2 + 9\alpha_1\alpha_2\alpha_3\beta_1\beta_2^2 + \alpha_1\alpha_2\alpha_3\beta_2^3 + \\ & \alpha_1\alpha_2\beta_1^3\beta_2 + 6\alpha_1\alpha_2\beta_1^2\beta_2^2 + \alpha_1\alpha_2\beta_1\beta_2^3 + \alpha_1\alpha_3^2\beta_1^2\beta_2 + \alpha_1\alpha_3^2\beta_1\beta_2^2 + \alpha_1\alpha_3\beta_1^3\beta_2 + 6\alpha_1\alpha_3\beta_1^2\beta_2^2 + \alpha_1\alpha_3\beta_1\beta_2^3 + \\ & \alpha_1\beta_1^3\beta_2^2 + \alpha_1\beta_1^2\beta_2^3 + \alpha_2^2\alpha_3^2\beta_1\beta_2 + \alpha_2^2\alpha_3\beta_1^2\beta_2 + \alpha_2^2\alpha_3\beta_1\beta_2^2 + \alpha_2^2\beta_1^2\beta_2^2 + \alpha_2\alpha_3^2\beta_1^2\beta_2 + \alpha_2\alpha_3^2\beta_1\beta_2^2 + \\ & \alpha_2\alpha_3\beta_1^3\beta_2 + 6\alpha_2\alpha_3\beta_1^2\beta_2^2 + \alpha_2\alpha_3\beta_1\beta_2^3 + \alpha_2\beta_1^3\beta_2^2 + \alpha_2\beta_1^2\beta_2^3 + \alpha_3^2\beta_1^2\beta_2^2 + \alpha_3\beta_1^3\beta_2^2 + \alpha_3\beta_1^2\beta_2^3 + \beta_1^3\beta_2^3 \end{aligned}$$

Observe that \mathbf{R} and \mathbf{S} have the same terms. Some terms appear a different number of times. For example, $\alpha_1^2\alpha_2\alpha_3\beta_1\beta_2$ appears two times in \mathbf{R} and six times in \mathbf{S} . However, since an commutative idempotent semiring ignores additive multiplicities, it does not matter. Hence we see that $\mathbf{R} = \mathbf{S}$.

4 Examples

In this section, we will show computational examples on *particular* CIS structures given in Example 1 of Section 2 to confirm the validity of the main result (Theorem 1) before its general proof (given in Section 6). We will use *structure-specific languages* whenever possible, in the hope that it would demonstrate the applicability of the main result in apparently very different contexts. We will confirm the main result via direct structure-specific computations. For easy computation, we consider only degree two polynomials.

Example 6 (Tropical). Let \mathcal{I} be the CIS of tropical semiring. Consider

$$f = \max(x, 1) + \max(x, 3), \quad g = \max(x, 2) + \max(x, 4)$$

We show that $\mathbf{R} = \mathbf{S}$ for the above f and g , via direct computations. Note

$$\begin{aligned}
\mathbf{R} &= \max(1, 2) + \max(1, 4) + \max(3, 2) + \max(3, 4) \\
&= 2 + 4 + 3 + 4 \\
&= 13 \\
\mathbf{S} &= \text{per} \begin{bmatrix} 0 & \max(1, 3) & 1 + 3 & -\infty \\ -\infty & 0 & \max(1, 3) & \max(1, 3) \\ 0 & \max(2, 4) & 2 + 4 & -\infty \\ -\infty & 0 & \max(2, 4) & 2 + 4 \end{bmatrix} \\
&= \text{per} \begin{bmatrix} 0 & 3 & 4 & -\infty \\ -\infty & 0 & 3 & 4 \\ 0 & 4 & 6 & -\infty \\ -\infty & 0 & 4 & 6 \end{bmatrix} \\
&= \max(4 + 6, 3 + 4 + 4, 3 + 4 + 6, 2 \cdot 4, 2 \cdot 6, 4 + 2 \cdot 4, 2 \cdot 3 + 6) \\
&= \max(10, 11, 13, 8, 12, 12, 12) \\
&= 13
\end{aligned}$$

Thus we have $\mathbf{R} = \mathbf{S}$.

Example 7 (Power set). Let \mathcal{I} be the CIS of the power set of \mathbb{R} . Consider

$$f = (x \cup [1, 2]) \cap (x \cup [3, 4]), \quad g = (x \cup [2, 3]) \cap (x \cup [4, 5])$$

We show that $\mathbf{R} = \mathbf{S}$ for the above f and g , via direct computations. Note

$$\begin{aligned}
\mathbf{R} &= ([1, 2] \cup [2, 3]) \cap ([1, 2] \cup [4, 5]) \cap ([3, 4] \cup [2, 3]) \cap ([3, 4] \cup [4, 5]) \\
&= [1, 3] \cap ([1, 2] \cup [4, 5]) \cap [2, 4] \cap [3, 5] \\
&= [1, 3] \cap ([1, 2] \cup [4, 5]) \cap [3, 4] \\
&= [3, 3] \cap ([1, 2] \cup [4, 5]) \\
&= \emptyset
\end{aligned}$$

$$\begin{aligned}
\mathbf{S} &= \text{per} \begin{bmatrix} \mathbb{R} & [1, 2] \cup [3, 4] & [1, 2] \cap [3, 4] & \emptyset \\ \emptyset & \mathbb{R} & [1, 2] \cup [3, 4] & [1, 2] \cap [3, 4] \\ \mathbb{R} & [2, 3] \cup [4, 5] & [2, 3] \cap [4, 5] & \emptyset \\ \emptyset & \mathbb{R} & [2, 3] \cup [4, 5] & [2, 3] \cap [4, 5] \end{bmatrix} \\
&= \text{per} \begin{bmatrix} \mathbb{R} & [1, 2] \cup [3, 4] & \emptyset & \emptyset \\ \emptyset & \mathbb{R} & [1, 2] \cup [3, 4] & \emptyset \\ \mathbb{R} & [2, 3] \cup [4, 5] & \emptyset & \emptyset \\ \emptyset & \mathbb{R} & [2, 3] \cup [4, 5] & \emptyset \end{bmatrix} \\
&= \emptyset
\end{aligned}$$

Thus we have $\mathbf{R} = \mathbf{S}$.

Example 8 (Topology). Let \mathcal{I} be a CIS of topology. In particular, we consider the cofinite topology on \mathbb{R} ; that is, open subsets are the empty set and the complementary of finite subsets of \mathbb{R} . This CIS is, of course, a sub-CIS of the power set CIS of \mathbb{R} in the previous example. But it is still interesting to consider, due to its importance. Consider

$$f = (x \cup (\mathbb{R} \setminus \{0, 1\})) \cap (x \cup (\mathbb{R} \setminus \{0, 2\})), \quad g = (x \cup (\mathbb{R} \setminus \{0, -1\})) \cap (x \cup (\mathbb{R} \setminus \{0, -2\}))$$

We show that $\mathbf{R} = \mathbf{S}$ for the above f and g , via direct computations. Note

$$\begin{aligned}
\mathbf{R} &= ((\mathbb{R} \setminus \{0, 1\}) \cup (\mathbb{R} \setminus \{0, -1\})) \cap ((\mathbb{R} \setminus \{0, 1\}) \cup (\mathbb{R} \setminus \{0, -2\})) \cap \\
&\quad ((\mathbb{R} \setminus \{0, 2\}) \cup (\mathbb{R} \setminus \{0, -1\})) \cap ((\mathbb{R} \setminus \{0, 2\}) \cup (\mathbb{R} \setminus \{0, -2\})) \\
&= \mathbb{R} \setminus \{0\} \\
\mathbf{S} &= \text{per} \begin{bmatrix} \mathbb{R} & (\mathbb{R} \setminus \{0, 1\}) \cup (\mathbb{R} \setminus \{0, 2\}) & (\mathbb{R} \setminus \{0, 1\}) \cap (\mathbb{R} \setminus \{0, 2\}) & \emptyset \\ \emptyset & \mathbb{R} & (\mathbb{R} \setminus \{0, 1\}) \cup (\mathbb{R} \setminus \{0, 2\}) & (\mathbb{R} \setminus \{0, 1\}) \cap (\mathbb{R} \setminus \{0, 2\}) \\ \mathbb{R} & (\mathbb{R} \setminus \{0, -1\}) \cup (\mathbb{R} \setminus \{0, -2\}) & (\mathbb{R} \setminus \{0, -1\}) \cap (\mathbb{R} \setminus \{0, -2\}) & \emptyset \\ \emptyset & \mathbb{R} & (\mathbb{R} \setminus \{0, -1\}) \cup (\mathbb{R} \setminus \{0, -2\}) & (\mathbb{R} \setminus \{0, -1\}) \cap (\mathbb{R} \setminus \{0, -2\}) \end{bmatrix} \\
&= \text{per} \begin{bmatrix} \mathbb{R} & \mathbb{R} \setminus \{0\} & \mathbb{R} \setminus \{0, 1, 2\} & \emptyset \\ \emptyset & \mathbb{R} & \mathbb{R} \setminus \{0\} & \mathbb{R} \setminus \{0, 1, 2\} \\ \mathbb{R} & \mathbb{R} \setminus \{0\} & \mathbb{R} \setminus \{0, -1, -2\} & \emptyset \\ \emptyset & \mathbb{R} & \mathbb{R} \setminus \{0\} & \mathbb{R} \setminus \{0, -1, -2\} \end{bmatrix} \\
&= (\mathbb{R} \setminus \{0\})^2 \cup ((\mathbb{R} \setminus \{0\})^2 \cap (\mathbb{R} \setminus \{0, -1, -2\}) \cup ((\mathbb{R} \setminus \{0\})^2 \cap (\mathbb{R} \setminus \{0, 1, 2\})) \cup \\
&\quad (\mathbb{R} \setminus \{0, -1, -2\})^2 \cup ((\mathbb{R} \setminus \{0, -1, -2\})^2 \cap (\mathbb{R} \setminus \{0, 1, 2\})) \cup (\mathbb{R} \setminus \{0, 1, 2\})^2 \\
&= (\mathbb{R} \setminus \{0\}) \cup (\mathbb{R} \setminus \{0, -1, -2\}) \cup (\mathbb{R} \setminus \{0, 1, 2\}) \cup (\mathbb{R} \setminus \{0, -1, -2\}) \cup ((\mathbb{R} \setminus \{0, -1, -2, 1, 2\}) \cup (\mathbb{R} \setminus \{0, 1, 2\})) \\
&= \mathbb{R} \setminus \{0\}
\end{aligned}$$

Thus we have $\mathbf{R} = \mathbf{S}$.

Example 9 (Compact-convex). Let \mathcal{I} be the CIS of the set of all compact convex subsets of \mathbb{R}^2 . Let $\langle x_1, \dots, x_r \rangle$ denote the convex hull of the points $x_1, \dots, x_r \in \mathbb{R}^2$. Consider

$$f = \left(x + \langle (0, 0) \rangle \right) \cdot \left(x + \langle (0, 0), (1, 0) \rangle \right), \quad g = \left(x + \langle (0, 0) \rangle \right) \cdot \left(x + \langle (0, 0), (0, 1) \rangle \right).$$

We show that $\mathbf{R} = \mathbf{S}$ for the above f and g , via direct computations. Note

$$\begin{aligned}
\mathbf{R} &= \left(\langle (0, 0) \rangle + \langle (0, 0) \rangle \right) \cdot \left(\langle (0, 0) \rangle + \langle (0, 0), (0, 1) \rangle \right) \cdot \left(\langle (0, 0), (1, 0) \rangle + \langle (0, 0) \rangle \right) \cdot \left(\langle (0, 0), (1, 0) \rangle + \langle (0, 0), (0, 1) \rangle \right) \\
&= \langle (0, 0) \rangle \cdot \langle (0, 0), (0, 1) \rangle \cdot \langle (0, 0), (1, 0) \rangle \cdot \langle (0, 0), (0, 1), (1, 0) \rangle \\
&= \langle (0, 0), (0, 2), (1, 2), (2, 0), (2, 1) \rangle \\
\mathbf{S} &= \text{per} \begin{bmatrix} \langle (0, 0) \rangle & \langle (0, 0) \rangle + \langle (0, 0), (1, 0) \rangle & \langle (0, 0) \rangle \cdot \langle (0, 0), (1, 0) \rangle & \emptyset \\ \emptyset & \langle (0, 0) \rangle & \langle (0, 0) \rangle + \langle (0, 0), (1, 0) \rangle & \langle (0, 0) \rangle \cdot \langle (0, 0), (1, 0) \rangle \\ \langle (0, 0) \rangle & \langle (0, 0) \rangle + \langle (0, 0), (0, 1) \rangle & \langle (0, 0) \rangle \cdot \langle (0, 0), (0, 1) \rangle & \emptyset \\ \emptyset & \langle (0, 0) \rangle & \langle (0, 0) \rangle + \langle (0, 0), (0, 1) \rangle & \langle (0, 0) \rangle \cdot \langle (0, 0), (0, 1) \rangle \end{bmatrix} \\
&= \text{per} \begin{bmatrix} \langle (0, 0) \rangle & \langle (0, 0), (1, 0) \rangle & \langle (0, 0), (1, 0) \rangle & \emptyset \\ \emptyset & \langle (0, 0) \rangle & \langle (0, 0), (1, 0) \rangle & \langle (0, 0), (1, 0) \rangle \\ \langle (0, 0) \rangle & \langle (0, 0), (0, 1) \rangle & \langle (0, 0), (0, 1) \rangle & \emptyset \\ \emptyset & \langle (0, 0) \rangle & \langle (0, 0), (0, 1) \rangle & \langle (0, 0), (0, 1) \rangle \end{bmatrix} \\
&= \langle (0, 0), (1, 0) \rangle \cdot \langle (0, 0), (0, 1) \rangle + \langle (0, 0), (1, 0) \rangle^2 + \langle (0, 0), (0, 1) \rangle^2 + \\
&\quad \langle (0, 0), (1, 0) \rangle \cdot \langle (0, 0), (0, 1) \rangle^2 + \langle (0, 0), (1, 0) \rangle^2 \cdot \langle (0, 0), (0, 1) \rangle \\
&= \langle (0, 0), (0, 1), (1, 0), (1, 1) \rangle + \langle (0, 0), (2, 0) \rangle + \langle (0, 0), (0, 1), (0, 2) \rangle + \\
&\quad \langle (0, 0), (0, 2), (1, 0), (1, 2) \rangle + \langle (0, 0), (0, 1), (2, 0), (2, 1) \rangle \\
&= \langle (0, 0), (0, 2), (1, 2), (2, 0), (2, 1) \rangle
\end{aligned}$$

Thus we have $\mathbf{R} = \mathbf{S}$.

Example 10 (Sequences). Let \mathcal{I} be the CIS of the set of all sequences over the boolean algebra $\{T, F\}$. Consider

$$f = \left(x + (T, T, F, \dots) \right) \left(x + (T, T, T, F, \dots) \right), \quad g = \left(x + (T, T, T, T, F, \dots) \right) \left(x + (T, T, T, T, T, F, \dots) \right)$$

In order to simplify the presentation, we will use the following short-hands. Let s_i denote the sequence (T, \dots, T, F, \dots) where T appears $i + 1$ times. Then we can write f and g succinctly as

$$f = (x + s_1)(x + s_2), \quad g = (x + s_3)(x + s_4).$$

Note that s_{-1} is the additive identity and s_0 is the multiplicative identity. Note also that $s_i + s_j = s_{\max\{i, j\}}$ and $s_i s_j = s_{i+j}$ for $i, j \geq 0$.

We show that $\mathbf{R} = \mathbf{S}$ for the above f and g , via direct computations. Note

$$\begin{aligned} \mathbf{R} &= (s_1 + s_3)(s_1 + s_4)(s_2 + s_3)(s_2 + s_4) \\ &= s_3 s_4 s_3 s_4 \\ &= s_{14} \\ &= (T, \dots, T, F, \dots) \quad \text{where } T \text{ is repeated 15 times} \end{aligned}$$

$$\begin{aligned} \mathbf{S} &= \text{per} \begin{bmatrix} s_0 & s_1 + s_2 & s_1 s_2 & s_{-1} \\ s_{-1} & s_0 & s_1 + s_2 & s_1 s_2 \\ s_0 & s_3 + s_4 & s_3 s_4 & s_{-1} \\ s_{-1} & s_0 & s_3 + s_4 & s_3 s_4 \end{bmatrix} \\ &= \text{per} \begin{bmatrix} s_0 & s_2 & s_3 & s_{-1} \\ s_{-1} & s_0 & s_2 & s_3 \\ s_0 & s_4 & s_7 & s_{-1} \\ s_{-1} & s_0 & s_4 & s_7 \end{bmatrix} \\ &= s_2^2 s_7 + s_2 s_3 s_4 + s_2 s_4 s_7 + s_3 s_4^2 + s_3^2 + s_3 s_7 + s_7^2 \\ &= s_{11} + s_9 + s_{13} + s_{11} + s_6 + s_{10} + s_{14} \\ &= s_{14} \\ &= (T, \dots, T, F, \dots) \quad \text{where } T \text{ is repeated 15 times} \end{aligned}$$

Thus we have $\mathbf{R} = \mathbf{S}$.

Example 11 (Ideals). Let \mathcal{I} be the CIS of the set of all the ideals of $\mathbb{C}[v_1, v_2]$. Consider

$$f = \left(x + \langle v_1^2 + v_2^2 - 1^2 \rangle \right) \left(x + \langle v_1^2 + v_2^2 - 2^2 \rangle \right), \quad g = \left(x + \langle v_1^2 - v_2^2 - 1^2 \rangle \right) \left(x + \langle v_1^2 - v_2^2 - 2^2 \rangle \right)$$

In order to simplify the presentation, we will use the following short-hands.

$$I_i = \langle v_1^2 + v_2^2 - i^2 \rangle, \quad J_j = \langle v_1^2 - v_2^2 - j^2 \rangle$$

Then we can write f and g succinctly as

$$f = (x + I_1)(x + I_2), \quad g = (x + J_1)(x + J_2)$$

We show that $\mathbf{R} = \mathbf{S}$ for the particular f and g , via direct computations. Note

$$\begin{aligned}
\mathbf{R} &= (I_1 + J_1)(I_1 + J_2)(I_2 + J_1)(I_2 + J_2) \\
\mathbf{S} &= \text{per} \begin{bmatrix} \langle 1 \rangle & I_1 + I_2 & I_1 I_2 & \{0\} \\ \{0\} & \langle 1 \rangle & I_1 + I_2 & I_1 I_2 \\ \langle 1 \rangle & J_1 + J_2 & J_1 J_2 & \{0\} \\ \{0\} & \langle 1 \rangle & J_1 + J_2 & J_1 J_2 \end{bmatrix} \\
&= \text{per} \begin{bmatrix} \langle 1 \rangle & \langle 1 \rangle & I_1 I_2 & \{0\} \\ \{0\} & \langle 1 \rangle & \langle 1 \rangle & I_1 I_2 \\ \langle 1 \rangle & \langle 1 \rangle & J_1 J_2 & \{0\} \\ \{0\} & \langle 1 \rangle & \langle 1 \rangle & J_1 J_2 \end{bmatrix} \\
&= (I_1 I_2)^2 + (J_1 J_2)^2 + (I_1 I_2)(J_1 J_2) + I_1 I_2 + J_1 J_2
\end{aligned}$$

After carrying out ideal additions and multiplications in a straightforward manner [1, 9, 22], we obtain

$$\begin{aligned}
\mathbf{R} = & \langle v_1^8 - 2v_1^4 v_2^4 + v_2^8 - 10v_1^6 + 10v_1^2 v_2^4 + 33v_1^4 - 17v_2^4 - 40v_1^2 + 16, v_1^8 - 2v_1^4 v_2^4 + v_2^8 - \\
& 13v_1^6 - 3v_1^4 v_2^2 + 13v_1^2 v_2^4 + 3v_2^6 + 60v_1^4 + 24v_1^2 v_2^2 - 20v_2^4 - 112v_1^2 - 48v_2^2 + 64, v_1^8 - 2v_1^4 v_2^4 + v_2^8 - \\
& 13v_1^6 + 3v_1^4 v_2^2 + 13v_1^2 v_2^4 - 3v_2^6 + 60v_1^4 - 24v_1^2 v_2^2 - 20v_2^4 - 112v_1^2 + 48v_2^2 + 64, v_1^8 - 2v_1^4 v_2^4 + v_2^8 - \\
& v_2^8 - 7v_1^6 - 3v_1^4 v_2^2 + 7v_1^2 v_2^4 + 3v_2^6 + 15v_1^4 + 6v_1^2 v_2^2 - 5v_2^4 - 13v_1^2 - 3v_2^2 + 4, v_1^8 - 2v_1^4 v_2^4 + v_2^8 - \\
& 7v_1^6 + 3v_1^4 v_2^2 + 7v_1^2 v_2^4 - 3v_2^6 + 15v_1^4 - 6v_1^2 v_2^2 - 5v_2^4 - 13v_1^2 + 3v_2^2 + 4, v_1^8 - 2v_1^6 v_2^2 + 2v_1^2 v_2^6 - \\
& v_2^8 - 13v_1^6 + 21v_1^4 v_2^2 - 3v_1^2 v_2^4 - 5v_2^6 + 60v_1^4 - 72v_1^2 v_2^2 + 12v_2^4 - 112v_1^2 + 80v_2^2 + 64, v_1^8 - 2v_1^6 v_2^2 + \\
& 2v_1^2 v_2^6 - v_2^8 - 10v_1^6 + 12v_1^4 v_2^2 + 6v_1^2 v_2^4 - 8v_2^6 + 33v_1^4 - 18v_1^2 v_2^2 - 15v_2^4 - 40v_1^2 + 8v_2^2 + 16, v_1^8 - \\
& 2v_1^6 v_2^2 + 2v_1^2 v_2^6 - v_2^8 - 10v_1^6 + 18v_1^4 v_2^2 - 6v_1^2 v_2^4 - 2v_2^6 + 33v_1^4 - 48v_1^2 v_2^2 + 15v_2^4 - 40v_1^2 + 32v_2^2 + \\
& 16, v_1^8 - 2v_1^6 v_2^2 + 2v_1^2 v_2^6 - v_2^8 - 7v_1^6 + 9v_1^4 v_2^2 + 3v_1^2 v_2^4 - 5v_2^6 + 15v_1^4 - 12v_1^2 v_2^2 - 3v_2^4 - 13v_1^2 + \\
& 5v_2^2 + 4, v_1^8 + 2v_1^6 v_2^2 - 2v_1^2 v_2^6 - v_2^8 - 13v_1^6 - 21v_1^4 v_2^2 - 3v_1^2 v_2^4 + 5v_2^6 + 60v_1^4 + 72v_1^2 v_2^2 + 12v_2^4 - \\
& 112v_1^2 - 80v_2^2 + 64, v_1^8 + 2v_1^6 v_2^2 - 2v_1^2 v_2^6 - v_2^8 - 10v_1^6 - 18v_1^4 v_2^2 - 6v_1^2 v_2^4 + 2v_2^6 + 33v_1^4 + 48v_1^2 v_2^2 + \\
& 15v_2^4 - 40v_1^2 - 32v_2^2 + 16, v_1^8 + 2v_1^6 v_2^2 - 2v_1^2 v_2^6 - v_2^8 - 10v_1^6 - 12v_1^4 v_2^2 + 6v_1^2 v_2^4 + 8v_2^6 + 33v_1^4 + \\
& 18v_1^2 v_2^2 - 15v_2^4 - 40v_1^2 - 8v_2^2 + 16, v_1^8 + 2v_1^6 v_2^2 - 2v_1^2 v_2^6 - v_2^8 - 7v_1^6 - 9v_1^4 v_2^2 + 3v_1^2 v_2^4 + 5v_2^6 + \\
& 15v_1^4 + 12v_1^2 v_2^2 - 3v_2^4 - 13v_1^2 - 5v_2^2 + 4, v_1^8 - 4v_1^6 v_2^2 + 6v_1^4 v_2^4 - 4v_1^2 v_2^6 + v_2^8 - 10v_1^6 + 30v_1^4 v_2^2 - \\
& 30v_1^2 v_2^4 + 10v_2^6 + 33v_1^4 - 66v_1^2 v_2^2 + 33v_2^4 - 40v_1^2 + 40v_2^2 + 16, v_1^8 + 4v_1^6 v_2^2 + 6v_1^4 v_2^4 + 4v_1^2 v_2^6 + \\
& v_2^8 - 10v_1^6 - 30v_1^4 v_2^2 - 30v_1^2 v_2^4 - 10v_2^6 + 33v_1^4 + 66v_1^2 v_2^2 + 33v_2^4 - 40v_1^2 - 40v_2^2 + 16 \rangle
\end{aligned}$$

$$\begin{aligned}
\mathbf{S} = & \langle v_1^4 - 2v_1^2 v_2^2 + v_2^4 - 5v_1^2 + 5v_2^2 + 4, v_1^4 + 2v_1^2 v_2^2 + v_2^4 - 5v_1^2 - 5v_2^2 + 4, v_1^8 - 2v_1^4 v_2^4 + \\
& v_2^8 - 10v_1^6 + 10v_1^2 v_2^4 + 33v_1^4 - 17v_2^4 - 40v_1^2 + 16, v_1^8 - 4v_1^6 v_2^2 + 6v_1^4 v_2^4 - 4v_1^2 v_2^6 + v_2^8 - 10v_1^6 + \\
& 30v_1^4 v_2^2 - 30v_1^2 v_2^4 + 10v_2^6 + 33v_1^4 - 66v_1^2 v_2^2 + 33v_2^4 - 40v_1^2 + 40v_2^2 + 16, v_1^8 + 4v_1^6 v_2^2 + 6v_1^4 v_2^4 + \\
& 4v_1^2 v_2^6 + v_2^8 - 10v_1^6 - 30v_1^4 v_2^2 - 30v_1^2 v_2^4 - 10v_2^6 + 33v_1^4 + 66v_1^2 v_2^2 + 33v_2^4 - 40v_1^2 - 40v_2^2 + 16 \rangle
\end{aligned}$$

Note that the generators of the two ideals look very different. However, after computing the reduced Gröbner basis [6, 7, 9] of the generators with respect to the total degree order ($v_1 \succ v_2$), we obtain

$$\begin{aligned}
\mathbf{R} &= \langle 2v_1^2 v_2^2 - 5v_2^2, v_1^4 + v_2^4 - 5v_1^2 + 4, 4v_2^6 - 9v_2^2 \rangle \\
\mathbf{S} &= \langle 2v_1^2 v_2^2 - 5v_2^2, v_1^4 + v_2^4 - 5v_1^2 + 4, 4v_2^6 - 9v_2^2 \rangle
\end{aligned}$$

Thus we have $\mathbf{R} = \mathbf{S}$.

5 Sketchy Proof of Main Result

The complete proof of the main result (Theorem 1) is long and technical. Hence, before we plunge into the complete proof (given in the next section), we informally sketch the overall structure/underlying ideas of the proof by using a small case $m = 5, n = 4$.

Note that

$$\mathbf{R} = \begin{pmatrix} (\alpha_1 + \beta_1) & (\alpha_1 + \beta_2) & (\alpha_1 + \beta_3) & (\alpha_1 + \beta_4) \\ (\alpha_2 + \beta_1) & (\alpha_2 + \beta_2) & (\alpha_2 + \beta_3) & (\alpha_2 + \beta_4) \\ (\alpha_3 + \beta_1) & (\alpha_3 + \beta_2) & (\alpha_3 + \beta_3) & (\alpha_3 + \beta_4) \\ (\alpha_4 + \beta_1) & (\alpha_4 + \beta_2) & (\alpha_4 + \beta_3) & (\alpha_4 + \beta_4) \\ (\alpha_5 + \beta_1) & (\alpha_5 + \beta_2) & (\alpha_5 + \beta_3) & (\alpha_5 + \beta_4) \end{pmatrix}$$

$$\mathbf{S} = \text{per} \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & & & \\ & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & & \\ & & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & \\ & & & a_0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ b_0 & b_1 & b_2 & b_3 & b_4 & & & & \\ & b_0 & b_1 & b_2 & b_3 & b_4 & & & \\ & & b_0 & b_1 & b_2 & b_3 & b_4 & & \\ & & & b_0 & b_1 & b_2 & b_3 & b_4 & \\ & & & & b_0 & b_1 & b_2 & b_3 & b_4 \end{bmatrix}$$

Recall that the main theorem (Theorem 1) states that $\mathbf{R} = \mathbf{S}$, that is, a term appears in \mathbf{R} if and only if it appears in \mathbf{S} . The overall strategy for the proof is to divide the task into showing the following four claims:

1. A term occurs in \mathbf{R} iff it has a representation in terms of a certain boolean matrix, which we call “res”-representation.
2. A term occurs in \mathbf{S} iff it has a representation in terms of a certain pair of boolean matrices, which we call “syl”-representation.
3. If a term has a res-representation then it has a syl-representation.
4. If a term has a syl-representation then it has a res-representation.

The main result (Theorem 1) immediately follows from the above four claims. In the following four subsections, we informally explain/justify the claims one by one.

5.1 A term occurs in \mathbf{R} iff it has a res-representation.

By expanding \mathbf{R} , we obtain

$$\mathbf{R} = \cdots + \alpha_1^3 \alpha_2^3 \alpha_3^3 \alpha_4^2 \alpha_5^2 \beta_1^2 \beta_2^2 \beta_3^2 \beta_4^1 + \cdots$$

The particular term above can be obtained by making the underlined choice while expanding \mathbf{R} .

$$\begin{pmatrix} (\underline{\alpha_1} + \beta_1) & (\underline{\alpha_1} + \beta_2) & (\underline{\alpha_1} + \beta_3) & (\underline{\alpha_1} + \beta_4) \\ (\underline{\alpha_2} + \beta_1) & (\underline{\alpha_2} + \beta_2) & (\underline{\alpha_2} + \beta_3) & (\underline{\alpha_2} + \beta_4) \\ (\underline{\alpha_3} + \beta_1) & (\underline{\alpha_3} + \beta_2) & (\underline{\alpha_3} + \beta_3) & (\underline{\alpha_3} + \beta_4) \\ (\underline{\alpha_4} + \beta_1) & (\underline{\alpha_4} + \beta_2) & (\underline{\alpha_4} + \beta_3) & (\underline{\alpha_4} + \beta_4) \\ (\underline{\alpha_5} + \beta_1) & (\underline{\alpha_5} + \beta_2) & (\underline{\alpha_5} + \beta_3) & (\underline{\alpha_5} + \beta_4) \end{pmatrix}$$

It is convenient to represent the choice with the following boolean matrix

$$\mathcal{R} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

where 1 in the i -th row means that α_i is chosen and 0 in the j -th column means that β_j is chosen. Then obviously the particular term above can be written as

$$\alpha_1^3 \alpha_2^3 \alpha_3^3 \alpha_4^2 \alpha_5^2 \beta_1^2 \beta_2^2 \beta_3^2 \beta_4^1 = \alpha^{rs(\mathcal{R})} \beta^{cs(\bar{\mathcal{R}})}$$

where $\bar{\mathcal{R}}$ stands for boolean complement of \mathcal{R} , rs for row sum and cs for column sum. Considering all other terms in the same manner, we have

$$\mathbf{R} = \sum_{\mathcal{R} \in \{0,1\}^{5 \times 4}} \alpha^{rs(\mathcal{R})} \beta^{cs(\bar{\mathcal{R}})}$$

Let us call the matrix \mathcal{R} a “res”-representation of the corresponding term. In summary, we observe that a term occurs in \mathbf{R} iff it has a res-representation.

5.2 A term occurs in \mathbf{S} iff it has a syl-representation.

By expanding \mathbf{S} , we obtain

$$\mathbf{S} = \cdots + \alpha_1^3 \alpha_2^3 \alpha_3^3 \alpha_4^2 \alpha_5^2 \beta_1^2 \beta_2^2 \beta_3^2 \beta_4^1 + \cdots$$

The particular term above can be obtained by making the following choice (permutation path) while expanding \mathbf{S} .

	α_3							
					$\alpha_1 \alpha_2 \alpha_3 \alpha_5$			
							$\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5$	
						$\alpha_1 \alpha_2 \alpha_4$		
1								
		β_2						
				$\beta_1 \beta_3$				
			1					
								$\beta_1 \beta_2 \beta_3 \beta_4$

(*)

It is convenient to represent the choice with the following boolean matrices

$$\mathcal{S}_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \mathcal{S}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

where the j -th column of \mathcal{S}_1 encodes the α term in the j -th row of the top 4×9 submatrix of the above matrix (*) and the i -th row of \mathcal{S}_2 encodes the β term in the i -th row of the bottom 5×9 submatrix of the above matrix (*). Then the particular term above can be written as

$$\alpha_1^3 \alpha_2^3 \alpha_3^3 \alpha_4^2 \alpha_5^2 \beta_1^2 \beta_2^2 \beta_3^2 \beta_4^1 = \alpha^{rs(\mathcal{S}_1)} \beta^{cs(\mathcal{S}_2)}$$

Let $c = cs(\mathcal{S}_1)$ and $r = rs(\mathcal{S}_2)$. Then $c = (1, 4, 5, 3)$ and $r = (0, 1, 2, 0, 4)$. Observe

$$\begin{aligned} c_1 + 1, c_2 + 2, c_3 + 3, c_4 + 4, r_1 + 1, r_2 + 2, r_3 + 3, r_4 + 4, r_5 + 5 \\ = 2, 6, 8, 7, 1, 3, 5, 4, 9 \end{aligned}$$

which is exactly the permutation path (the choice of columns) yielding the particular term. This motivates the following short-hand notations:

$$\begin{aligned} acs(\mathcal{S}_1) &= (c_1 + 1, c_2 + 2, c_3 + 3, c_4 + 4) \\ ars(\mathcal{S}_2) &= (r_1 + 1, r_2 + 2, r_3 + 3, r_4 + 4, r_5 + 5) \end{aligned}$$

Using these notations, we can restate the above observation (property) as

$$\{c'_1, \dots, c'_4, r'_1, \dots, r'_5\} = \{1, \dots, 4 + 5\}$$

where $c' = \text{acs}(\mathcal{S}_1)$ and $r' = \text{arc}(\mathcal{S}_2)$. We will denote this property by $PC(\mathcal{S}_1, \mathcal{S}_2)$. Considering all other terms in the same manner, we have

$$\mathbf{R} = \sum_{\substack{\mathcal{S}_1, \mathcal{S}_2 \in \{0,1\}^{5 \times 4} \\ PC(\mathcal{S}_1, \mathcal{S}_2)}} \alpha^{rs(\mathcal{S}_1)} \beta^{cs(\mathcal{S}_2)}$$

Let us call the pair of the matrices $(\mathcal{S}_1, \mathcal{S}_2)$ a “syl”-representation of the corresponding term. In summary, we observe that a term occurs in \mathbf{S} iff it has a syl-representation.

5.3 If a term has a res-representation then it has a syl-representation.

Consider the term $t = \alpha_1^3 \alpha_2^3 \alpha_3^3 \alpha_4^2 \alpha_5^2 \beta_1^2 \beta_2^2 \beta_3^2 \beta_4^1$ in \mathbf{R} obtained by the following choice

$$\begin{pmatrix} \underline{\alpha_1} + \underline{\beta_1} \\ \underline{\alpha_2} + \underline{\beta_1} \\ \underline{\alpha_3} + \underline{\beta_1} \\ \underline{\alpha_4} + \underline{\beta_1} \\ \underline{\alpha_5} + \underline{\beta_1} \end{pmatrix} \begin{pmatrix} \underline{\alpha_1} + \underline{\beta_2} \\ \underline{\alpha_2} + \underline{\beta_2} \\ \underline{\alpha_3} + \underline{\beta_2} \\ \underline{\alpha_4} + \underline{\beta_2} \\ \underline{\alpha_5} + \underline{\beta_2} \end{pmatrix} \begin{pmatrix} \underline{\alpha_1} + \underline{\beta_3} \\ \underline{\alpha_2} + \underline{\beta_3} \\ \underline{\alpha_3} + \underline{\beta_3} \\ \underline{\alpha_4} + \underline{\beta_3} \\ \underline{\alpha_5} + \underline{\beta_3} \end{pmatrix} \begin{pmatrix} \underline{\alpha_1} + \underline{\beta_4} \\ \underline{\alpha_2} + \underline{\beta_4} \\ \underline{\alpha_3} + \underline{\beta_4} \\ \underline{\alpha_4} + \underline{\beta_4} \\ \underline{\alpha_5} + \underline{\beta_4} \end{pmatrix}$$

which is represented by the following res-representation

$$\mathcal{R} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

We construct the following two boolean matrices

$$\mathcal{S}_1 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \mathcal{S}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

where $\mathcal{S}_1 = \mathcal{R}$, $cs(\mathcal{S}_2) = cs(\bar{\mathcal{R}})$ and 1's in \mathcal{S}_2 are “flushed” to the bottom along columns. Observe that $(\mathcal{S}_1, \mathcal{S}_2)$ represents the following choice yielding the above term t , that is, it is a syl-representation of the term t .

			$\alpha_1 \alpha_2 \alpha_3$					
				$\alpha_2 \alpha_3 \alpha_5$				
					$\alpha_1 \alpha_4 \alpha_5$			
						$\alpha_1 \alpha_2 \alpha_3 \alpha_4$		
1								
	1							
		1						
						$\beta_1 \beta_2 \beta_3$		
								$\beta_1 \beta_2 \beta_3 \beta_4$

In summary, if a term has a res-representation then it has a syl-representation.

5.4 If a term has a syl-representation then it has a res-representation.

Consider the term $t = \alpha_1^3 \alpha_2^3 \alpha_3^3 \alpha_4^2 \alpha_5^2 \beta_1^2 \beta_2^2 \beta_3^2 \beta_4^1$ in \mathbf{S} obtained by the following choice

	α_3							
					$\alpha_1 \alpha_2 \alpha_3 \alpha_5$			
							$\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5$	
						$\alpha_1 \alpha_2 \alpha_4$		
1								
		β_2						
				$\beta_1 \beta_3$				
			1					
								$\beta_1 \beta_2 \beta_3 \beta_4$

which is represented by the following syl-representation

$$\mathcal{S}_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \mathcal{S}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

We repeatedly transform the syl-representations of the term t as follows.

1. We make $acs(\mathcal{S}_1)$ and $ars(\mathcal{S}_2)$ increasing.

By swapping

- $\mathcal{S}_{1,3,3}$ and $\mathcal{S}_{1,3,4}$
- $\mathcal{S}_{2,3,1}$ and $\mathcal{S}_{2,4,1}$

of the above syl-representation, we obtain another syl-representation

$$\mathcal{S}_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \mathcal{S}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

where $\mathcal{S}_{k,i,j}$ denotes the (i,j) entry of the matrix \mathcal{S}_k . It represents the following choice.

	α_3							
					$\alpha_1 \alpha_2 \alpha_3 \alpha_5$			
						$\alpha_1 \alpha_2 \alpha_4 \alpha_5$		
							$\alpha_1 \alpha_2 \alpha_3 \alpha_4$	
1								
		β_2						
			β_3					
				β_1				
								$\beta_1 \beta_2 \beta_3 \beta_4$

Note that the column index of the top/bottom part of the matrix is increasing as the row index is increasing.

2. We make 1's in \mathcal{S}_2 “flushed to bottom”.

By swapping

- $\mathcal{S}_{2,2,2}$ and $\mathcal{S}_{2,3,2}$
- $\mathcal{S}_{2,3,2}$ and $\mathcal{S}_{2,4,2}$
- $\mathcal{S}_{1,1,1}$ and $\mathcal{S}_{1,1,2}$
- $\mathcal{S}_{2,3,3}$ and $\mathcal{S}_{2,4,3}$
- $\mathcal{S}_{1,2,1}$ and $\mathcal{S}_{1,2,3}$

we obtain still another syl-representation

$$\mathcal{S}_1 = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \quad \mathcal{S}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

It represents the following choice.

			$\alpha_1\alpha_2\alpha_3$					
				$\alpha_2\alpha_3\alpha_5$				
					$\alpha_1\alpha_4\alpha_5$			
						$\alpha_1\alpha_2\alpha_3\alpha_4$		
1								
	1							
		1						
					$\beta_1\beta_2\beta_3$			
							$\beta_1\beta_2\beta_3\beta_4$	

Note that the entries in the bottom part of the matrix are products of the form $\beta_1 \cdots \beta_k$ (product of consecutive β 's starting from β_1).

Let $\mathcal{R} = \mathcal{S}_1$. Observe that \mathcal{R} represents the following choice yielding the given term t , that is, it is a res-representation of the term t .

$$\begin{pmatrix} \alpha_1 + \beta_1 \\ \alpha_2 + \beta_1 \\ \alpha_3 + \beta_1 \\ \alpha_4 + \beta_1 \\ \alpha_5 + \beta_1 \end{pmatrix} \begin{pmatrix} \alpha_1 + \beta_2 \\ \alpha_2 + \beta_2 \\ \alpha_3 + \beta_2 \\ \alpha_4 + \beta_2 \\ \alpha_5 + \beta_2 \end{pmatrix} \begin{pmatrix} \alpha_1 + \beta_3 \\ \alpha_2 + \beta_3 \\ \alpha_3 + \beta_3 \\ \alpha_4 + \beta_3 \\ \alpha_5 + \beta_3 \end{pmatrix} \begin{pmatrix} \alpha_1 + \beta_4 \\ \alpha_2 + \beta_4 \\ \alpha_3 + \beta_4 \\ \alpha_4 + \beta_4 \\ \alpha_5 + \beta_4 \end{pmatrix}$$

In summary, if a term has a syl-representation then it has a res-representation.

6 Detailed Proof of Main Result

In this section, we provide a detailed proof for the main result. We strongly recommend that the reader first go over the previous section (Section 5) where we provided an informal/intuitive sketch of the overall structure/underlying ideas of the proof. It will greatly aid the reader in following the detailed and technical proof given below.

Recall that the main theorem (Theorem 1) claims that $\mathbf{R} = \mathbf{S}$, that is, a term appears in \mathbf{R} if and only if it appears in \mathbf{S} . It follows immediately from the following four lemmas.

Lemma 2: A term occurs in \mathbf{R} iff it has a res-representation (a certain boolean matrix).

Lemma 3: A term occurs in \mathbf{S} iff it has a syl-representation (a certain pair of boolean matrices).

Lemma 7: If a term has a res-representation then it has a syl-representation.

Lemma 10: If a term has a syl-representation then it has a res-representation.

We will devote one subsection for each lemma. Each subsection ends with the proof of each of the above lemmas. Lemmas 7 and 10 are proved constructively, by providing algorithms (Algorithms 1 and 4) that produce one representation from the other. These algorithms are based on a key lemma (Lemma 6) that establishes a crucial relationship between the two representations (syl and res). Before stating and proving the above lemmas, we introduce notations that will be used throughout this section.

Notation 5. Let $M \in \{0, 1\}^{m \times n}$.

1. The complement of M , written as \bar{M} , is the $m \times n$ matrix defined by

$$\bar{M}_{ij} = 1 - M_{ij}$$

2. The row sum of M , written as $rs(M)$, is the m -dimensional vector defined by

$$rs(M) = \left(\sum_{j=1}^n M_{ij} \right)_{i=1, \dots, m}$$

3. The column sum of M , written as $cs(M)$, is the n -dimensional vector defined by

$$cs(M) = \left(\sum_{i=1}^m M_{ij} \right)_{j=1, \dots, n}$$

4. The adjusted row sum of M , written as $ars(M)$, is the m -dimensional vector defined by

$$ars(M) = \left(i + \sum_{j=1}^n M_{ij} \right)_{i=1, \dots, m}$$

5. The adjusted column sum of M , written as $acs(M)$, is the n -dimensional vector defined by

$$acs(M) = \left(j + \sum_{i=1}^m M_{ij} \right)_{j=1, \dots, n}$$

Example 12. Let

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Then

$$rs(M) = (3, 2, 2, 1)$$

$$cs(M) = (2, 3, 3)$$

$$ars(M) = (4, 4, 5, 5)$$

$$acs(M) = (3, 5, 6)$$

6.1 A term occurs in \mathbf{R} iff it has a res-representation.

Definition 6 (Res-representation). Let $t = \alpha^\mu \beta^\nu$ be a term, where $\mu \in \mathbb{Z}_{\geq 0}^m$ and $\nu \in \mathbb{Z}_{\geq 0}^n$. Let $\mathcal{R} \in \{0, 1\}^{m \times n}$. We say that \mathcal{R} is a res-representation of t if

- $rs(\mathcal{R}) = \mu$.
- $cs(\bar{\mathcal{R}}) = \nu$. Equivalently $cs(\mathcal{R}) = \bar{\nu}$ where $\bar{\nu}_j = m - \nu_j$.

Example 13. Let $m = 3$ and $n = 2$. Let $t = \alpha_1^2 \alpha_2^1 \alpha_3^1 \beta_1^1 \beta_2^1 = \alpha^{(2,1,1)} \beta^{(1,1)}$. Let

$$\mathcal{R} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We would like to know whether \mathcal{R} is a res-representation of t . Note

$$\mathcal{R} = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hline & & \mu_i \\ \hline 2 & 2 & \bar{\nu}_j \\ 1 & 1 & \nu_j \end{array} \right]$$

Hence \mathcal{R} is a res-representation of t . Likewise, the following matrix is also a res-representation of t .

$$\mathcal{R} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Lemma 2. A term occurs in \mathbf{R} iff it has a res-representation.

Proof. Recall

$$\mathbf{R} = \prod_{i=1}^m \prod_{j=1}^n (\alpha_i + \beta_j)$$

Note

$$\begin{aligned} \mathbf{R} &= \prod_{i=1}^m \prod_{j=1}^n \sum_{e \in \{0,1\}} \alpha_i^e \beta_j^{\bar{e}} \quad \text{where } \bar{e} = 1 - e \\ &= \sum_{\mathcal{R} \in \{0,1\}^{m \times n}} \prod_{i=1}^m \prod_{j=1}^n \alpha_i^{\mathcal{R}_{ij}} \beta_j^{\bar{\mathcal{R}}_{ij}} \\ &= \sum_{\mathcal{R} \in \{0,1\}^{m \times n}} \prod_{i=1}^m \prod_{j=1}^n \alpha_i^{\mathcal{R}_{ij}} \prod_{i=1}^m \prod_{j=1}^n \beta_j^{\bar{\mathcal{R}}_{ij}} \\ &= \sum_{\mathcal{R} \in \{0,1\}^{m \times n}} \prod_{i=1}^m \alpha_i^{\sum_{j=1}^n \mathcal{R}_{ij}} \prod_{j=1}^n \beta_j^{\sum_{i=1}^m \bar{\mathcal{R}}_{ij}} \\ &= \sum_{\mathcal{R} \in \{0,1\}^{m \times n}} \alpha^\mu \beta^\nu \quad \text{where } \mu = rs(\mathcal{R}) \text{ and } \nu = cs(\bar{\mathcal{R}}) \end{aligned}$$

The claim follows immediately. \square

6.2 A term occurs in S iff it has a syl-representation.

Definition 7 (Properly coupled). Let $\mathcal{S}_1, \mathcal{S}_2 \in \{0, 1\}^{m \times n}$. We say that \mathcal{S}_1 and \mathcal{S}_2 are properly coupled and write as $PC(\mathcal{S}_1, \mathcal{S}_2)$ iff

$$\{c'_1, \dots, c'_n, r'_1, \dots, r'_m\} = \{1, \dots, m+n\}$$

where $c' = \text{acs}(\mathcal{S}_1)$ and $r' = \text{ars}(\mathcal{S}_2)$.

Example 14. Let

$$\mathcal{S}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \mathcal{S}_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since

$$\begin{aligned} c' &= \text{acs}(\mathcal{S}_1) = (2, 5) \\ r' &= \text{ars}(\mathcal{S}_2) = (1, 3, 4) \end{aligned}$$

we have

$$\{2, 5, 1, 3, 4\} = \{1, 2, 3, 4, 5\}$$

Hence we have $PC(\mathcal{S}_1, \mathcal{S}_2)$.

Definition 8 (Syl-representation). Let $t = \alpha^\mu \beta^\nu$ be a term. Let $\mathcal{S}_1, \mathcal{S}_2 \in \{0, 1\}^{m \times n}$. We say that $(\mathcal{S}_1, \mathcal{S}_2)$ is a syl-representation of t if

- $rs(\mathcal{S}_1) = \mu$
- $cs(\mathcal{S}_2) = \nu$
- $PC(\mathcal{S}_1, \mathcal{S}_2)$

Example 15. Let $m = 3$ and $n = 2$. Let $t = \alpha_1^2 \alpha_2^1 \alpha_3^1 \beta_1^1 \beta_2^1 = \alpha^{(2,1,1)} \beta^{(1,1)}$. Let

$$\mathcal{S}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \quad \mathcal{S}_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We would like to know whether $(\mathcal{S}_1, \mathcal{S}_2)$ is a syl-representation of t . Note

$$\mathcal{S}_1 = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \\ \hline & & \mu_i \\ \hline 1 & 3 & c_j \\ 1 & 2 & j \\ 2 & 5 & c_j + j \end{array} \right] \quad \mathcal{S}_2 = \left[\begin{array}{cc|ccc} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 3 & 4 \\ \hline & & r_i & i & r_i + i \\ \hline 1 & 1 & \nu_j & & \end{array} \right]$$

Hence $(\mathcal{S}_1, \mathcal{S}_2)$ is a syl-representation of t . Likewise, the following pairs of matrices are also syl-representations of t .

$$\begin{aligned} \mathcal{S}_1 &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} & \mathcal{S}_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \\ \mathcal{S}_1 &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} & \mathcal{S}_2 &= \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
\mathcal{S}_1 &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} & \mathcal{S}_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \\
\mathcal{S}_1 &= \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} & \mathcal{S}_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \\
\mathcal{S}_1 &= \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} & \mathcal{S}_2 &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}
\end{aligned}$$

Lemma 3. *A term occurs in \mathbf{S} iff it has a syl-representation.*

Proof. Let M be the Sylvester matrix of f and g (see Definition 3). Then

$\mathbf{S} = \text{per}(M)$

$$\begin{aligned}
&= \sum_{\{\sigma_1, \dots, \sigma_{n+m}\} = \{1, \dots, n+m\}} \prod_{i=1}^{n+m} M_{i\sigma_i} \\
&= \sum_{\{\sigma_1, \dots, \sigma_n, \sigma_{n+1}, \dots, \sigma_{n+m}\} = \{1, \dots, n+m\}} \left(\prod_{j=1}^n M_{j\sigma_j} \right) \left(\prod_{i=1}^m M_{(n+i)\sigma_{n+i}} \right) \\
&= \sum_{\{\sigma_1, \dots, \sigma_n, \sigma_{n+1}, \dots, \sigma_{n+m}\} = \{1, \dots, n+m\}} \left(\prod_{j=1}^n a_{\sigma_j-j} \right) \left(\prod_{i=1}^m b_{\sigma_{n+i}-i} \right) \\
&= \sum_{\{c_1+1, \dots, c_n+n, r_1+1, \dots, r_m+m\} = \{1, \dots, n+m\}} \left(\prod_{j=1}^n a_{c_j} \right) \left(\prod_{i=1}^m b_{r_i} \right), \text{ by reindexing with } c_j = \sigma_j - j \text{ and } r_j = \sigma_{n+i} - i \\
&= \sum_{\{c_1+1, \dots, c_n+n, r_1+1, \dots, r_m+m\} = \{1, \dots, n+m\}} \left(\prod_{j=1}^n \sum_{\substack{S_1 \in \{0,1\}^n \\ \sum_{k=1}^n S_{1,k}=c_j}} \prod_{i=1}^m \alpha_i^{S_{1,i}} \right) \left(\prod_{i=1}^m \sum_{\substack{S_2 \in \{0,1\}^n \\ \sum_{k=1}^m S_{2,k}=r_j}} \prod_{j=1}^n \beta_j^{S_{2,j}} \right) \\
&= \sum_{\{c_1+1, \dots, c_n+n, r_1+1, \dots, r_m+m\} = \{1, \dots, n+m\}} \left(\sum_{\substack{S_1 \in \{0,1\}^{m \times n} \\ cs(S_1)=c}} \prod_{i=1}^m \prod_{j=1}^n \alpha_i^{S_{1,i,j}} \right) \left(\sum_{\substack{S_2 \in \{0,1\}^{m \times n} \\ rs(S_2)=r}} \prod_{i=1}^m \prod_{j=1}^n \beta_j^{S_{2,i,j}} \right) \\
&= \sum_{\substack{S_1, S_2 \in \{0,1\}^{m \times n} \\ cs(S_1)=c \\ rs(S_2)=r \\ \{c_1+1, \dots, c_n+n, r_1+1, \dots, r_m+m\} = \{1, \dots, n+m\}}} \prod_{i=1}^m \prod_{j=1}^n \alpha_i^{S_{1,i,j}} \prod_{i=1}^m \prod_{j=1}^n \beta_j^{S_{2,i,j}} \\
&= \sum_{\substack{S_1, S_2 \in \{0,1\}^{m \times n} \\ PC(S_1, S_2)}} \prod_{i=1}^m \prod_{j=1}^n \alpha_i^{S_{1,i,j}} \prod_{j=1}^n \prod_{i=1}^m \beta_j^{S_{2,i,j}} \\
&= \sum_{\substack{S_1, S_2 \in \{0,1\}^{m \times n} \\ PC(S_1, S_2)}} \prod_{i=1}^m \alpha_i^{\sum_{j=1}^n S_{1,i,j}} \prod_{j=1}^n \beta_j^{\sum_{i=1}^m S_{2,i,j}} \\
&= \sum_{\substack{S_1, S_2 \in \{0,1\}^{m \times n} \\ PC(S_1, S_2)}} \alpha^\mu \beta^\nu \text{ where } \mu = rs(S_1) \text{ and } \nu = cs(S_2)
\end{aligned}$$

The claim follows immediately. \square

6.3 If a term has a res-representation then it has a syl-representation.

The proof is constructive, that is, it provides an algorithm that takes a res-representation and produces a syl-representation (Algorithm 1). The algorithm is immediate from a key lemma (Lemma 6) that establishes a crucial relationship between the two representations (syl and res). Thus most of this subsection will be devoted in stating and proving the key lemma.

Note that \mathbf{R} is a symmetric expression in $\alpha_1, \dots, \alpha_m$ and in β_1, \dots, β_n . Thus for any term in \mathbf{R} , every term obtained by permuting $\alpha_1, \dots, \alpha_m$ and permuting β_1, \dots, β_n is also in \mathbf{R} . The same holds for \mathbf{S} too. Hence, without loss of generality, we may restrict the proof to the terms $\alpha_1^{\mu_1} \dots \alpha_m^{\mu_m} \beta_1^{\nu_1} \dots \beta_n^{\nu_n}$ where $\mu_1 \geq \mu_2 \geq \dots \geq \mu_m$ and $\nu_1 \geq \nu_2 \geq \dots \geq \nu_n$. Therefore, from now on, we will assume that μ and ν are in non-increasing order.

Definition 9 (Bottom-left flushed). *A matrix is called bottom-left flushed if all the non-zero entries are flushed to the bottom-left. Let $c \in \{0, \dots, m\}^n$. Then the bottom-left flushed matrix of c , written as $F_c \in \{0, 1\}^{m \times n}$, is the bottom-left flushed matrix such that $cs(F_c) = c$.*

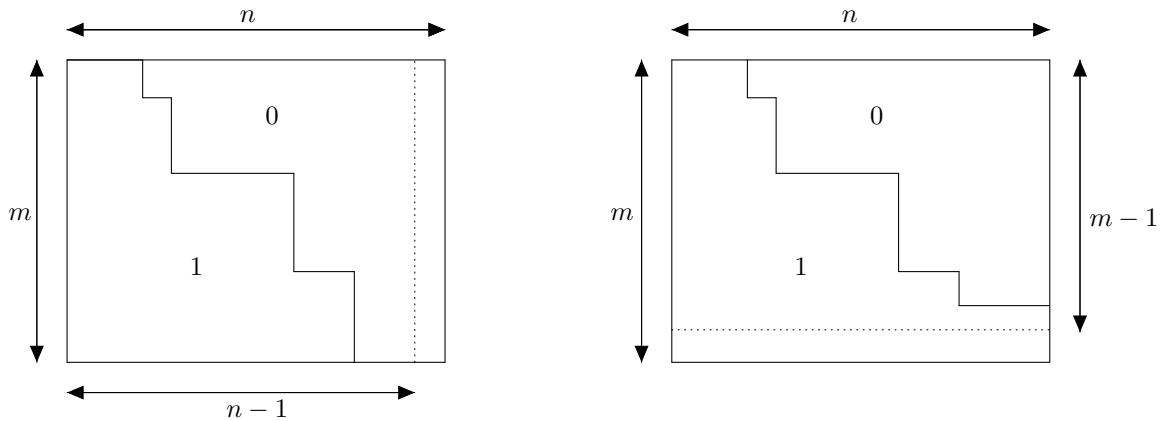
Example 16. *Let*

$$M = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Then M is flushed. Note also that $M = F_{(4,2,2,1,0)}$.

Lemma 4. *Let $M \in \{0, 1\}^{m \times n}$ be bottom-left flushed. Then we have $PC(\bar{M}, M)$.*

Proof. We will prove by mathematical induction on $m + n$. If $m + n = 0$ (i.e. $m = n = 0$), then the implication holds vacuously. Now, let us assume that the implication holds for all bottom-left flushed $m \times n$ matrix such that $m + n < k$. Consider an arbitrary bottom-left flushed $m \times n$ matrix M such that $m + n = k$. Let $c = cs(\bar{M})$ and $r = rs(M)$. We consider two cases as in the following two figures,



where we have all 0's above the "stairs" (jagged solid lines) and all 1's below the stairs.

Case $M_{mn} = 0$. Since M is bottom-left flushed, the last column of M is all zero like the above left figure. Let M^* be the $m \times (n-1)$ matrix obtained from M by deleting the last column of M . Note that M^* is also bottom-left flushed. Let $c^* = cs(\overline{M^*})$ and $r^* = rs(M^*)$. Then

$$r = r^* \text{ and } c = (c_1^*, \dots, c_{n-1}^*, m).$$

(In the above, c and c^* are the column sum vectors of \bar{M} and $\overline{M^*}$, respectively, and hence they count the number of 0 on the columns of M and M^*). Thus

$$\begin{aligned} & \{c_1 + 1, \dots, c_n + n, r_1 + 1, \dots, r_m + m\} \\ &= \{c_1^* + 1, \dots, c_{n-1}^* + n - 1, m + n, r_1^* + 1, \dots, r_m^* + m\} \\ &= \{c_1^* + 1, \dots, c_{n-1}^* + n - 1, r_1^* + 1, \dots, r_m^* + m\} \cup \{m + n\} \\ &= \{1, \dots, m + n - 1\} \cup \{m + n\} \quad \text{from the induction hypothesis} \\ &= \{1, \dots, m + n\}. \end{aligned}$$

Case $M_{mn} = 1$. Since M is bottom-left flushed, the last row of M is all one like the above right figure. Let M^* be the $(m-1) \times n$ matrix obtained from M by deleting the last row of M . Note that M^* is also bottom-left flushed. Let $c^* = cs(\overline{M^*})$ and $r^* = rs(M^*)$. Note that

$$c = c^* \text{ and } r = (r_1^*, \dots, r_{m-1}^*, n).$$

Thus

$$\begin{aligned} & \{c_1 + 1, \dots, c_n + n, r_1 + 1, \dots, r_m + m\} \\ &= \{c_1^* + 1, \dots, c_n^* + n, r_1^* + 1, \dots, r_{m-1}^* + m - 1, n + m\} \\ &= \{c_1^* + 1, \dots, c_n^* + n, r_1^* + 1, \dots, r_{m-1}^* + m - 1\} \cup \{n + m\} \\ &= \{1, \dots, m - 1 + n\} \cup \{n + m\} \quad \text{from the induction hypothesis} \\ &= \{1, \dots, m + n\}. \end{aligned}$$

Therefore, in both cases, $PC(\bar{M}, M)$ holds. □

Definition 10 (Sorted/Flushed). Let $A, B \in \{0, 1\}^{m \times n}$. We say that (A, B) is sorted iff $acs(A)$ and $ars(B)$ are sorted in increasing order. We say that (A, B) is flushed iff B is bottom-left flushed. We say that (A, B) is sorted-flushed iff it is both sorted and flushed.

Example 17. Let

$$(A, B) = \left[\begin{array}{ccc|c} 1 & 0 & 1 & \\ 0 & 1 & 1 & \\ 0 & 1 & 1 & \\ 1 & 0 & 0 & \\ \hline 2 & 2 & 3 & c_j \\ 3 & 4 & 6 & c_j + j \end{array} \right], \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 \\ 1 & 1 & 0 & 2 & 5 \\ 1 & 1 & 1 & 3 & 7 \\ \hline & & & r_i & r_i + i \end{array} \right]$$

Note that $acs(A) = (3, 4, 6)$ and $ars(B) = (1, 2, 5, 7)$ are sorted in increasing order. Thus (A, B) is sorted. Note that B is bottom-left flushed. Thus (A, B) is flushed. Hence (A, B) is sorted-flushed.

Lemma 5. Let $A, B \in \{0, 1\}^{m \times n}$. If (A, B) is sorted-flushed, then we have

$$cs(A) = cs(\bar{B}) \iff PC(A, B)$$

Proof. Assume that (A, B) is sorted-flushed. We need to show $cs(A) = cs(\bar{B}) \iff PC(A, B)$. We will show direction of implication one by one.

\Rightarrow Assume $cs(A) = cs(\bar{B})$. Then we have $acs(A) = acs(\bar{B})$. From Lemma 4, we have $PC(\bar{B}, B)$. Thus we have $PC(A, B)$.

\Leftarrow Assume $PC(A, B)$. Then we have

$$acs(A) = (1, \dots, m+n) \setminus ars(B).$$

From Lemma 4, we have $PC(\bar{B}, B)$. Thus we have

$$acs(\bar{B}) = (1, \dots, m+n) \setminus ars(B).$$

Thus $acs(A) = acs(\bar{B})$ and in turn $cs(A) = cs(\bar{B})$. □

Lemma 6. Let $t = \alpha^\mu \beta^\nu$ be a term. Let $\mathcal{R} \in \{0, 1\}^{m \times n}$. The following two are equivalent.

(1) \mathcal{R} is a res-representation of t .

(2) (\mathcal{R}, F_ν) is a sorted-flushed syl-representation of t .

Proof. We show each direction of implication one by one.

(1) \Rightarrow (2). It follows immediately from the following claims:

C1: (\mathcal{R}, F_ν) is sorted-flushed.

From the definition of F_ν , it is obvious that $ars(F_\nu)$ is sorted in increasing order and that F_ν is bottom-left flushed. Thus it remains to show that $acs(\mathcal{R})$ is sorted in increasing order. Since \mathcal{R} is a res-representation of t , we have $cs(\mathcal{R}) = \bar{\nu}$. Recall that at the very beginning of this section we assumed, without loss of generality, that ν is sorted in non-increasing order. So, $cs(\mathcal{R})$ is sorted in non-decreasing order. Thus, $acs(\mathcal{R})$ is sorted in increasing order.

C2: (\mathcal{R}, F_ν) is a syl-representation of t .

Since \mathcal{R} is a res-representation of t , we have $rs(\mathcal{R}) = \mu$. From the definitions of F_ν , we have $cs(F_\nu) = \nu$. Thus it remains to show that $PC(\mathcal{R}, F_\nu)$. Note

$$cs(\mathcal{R}) = \bar{\nu} = \overline{cs(F_\nu)} = cs(\overline{F_\nu}).$$

Therefore, from C1 and Lemma 5, we have $PC(\mathcal{R}, F_\nu)$.

(2) \Rightarrow (1). Since (\mathcal{R}, F_ν) is a syl-representation of t , we have $rs(\mathcal{R}) = \mu$. Thus it remains to show that $cs(\mathcal{R}) = \bar{\nu}$. Since (\mathcal{R}, F_ν) is a sorted-flushed syl-representation of t , we have that (\mathcal{R}, F_ν) is sorted-flushed and $PC(\mathcal{R}, F_\nu)$. Thus, from Lemma 5, we have

$$cs(\mathcal{R}) = cs(\overline{F_\nu}) = \overline{cs(F_\nu)} = \bar{\nu}.$$

□

Algorithm 1 (*SylFromRes*).

In: \mathcal{R} , a res-representation of a term t

Out: $(\mathcal{S}_1, \mathcal{S}_2)$, a syl-representation of the term t

$c \leftarrow cs(\mathcal{R})$

$(\mathcal{S}_1, \mathcal{S}_2) \leftarrow (\mathcal{R}, F_{\bar{c}})$

return $(\mathcal{S}_1, \mathcal{S}_2)$

Example 18. We trace the algorithm *SylFromRes* on the following input.

$$\text{In: } \mathcal{R} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

which is a res-representation of the term $t = \alpha_1^3 \alpha_2^3 \alpha_3^3 \alpha_4^2 \alpha_5^2 \beta_1^2 \beta_2^2 \beta_3^2 \beta_4^1$.

$$c = [3, 3, 3, 4]$$

$$F_{\bar{c}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\text{Out: } (\mathcal{S}_1, \mathcal{S}_2) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

which is a syl-representation of the term t .

Lemma 7. The algorithm 1 (*SylFromRes*) is correct. Thus if a term has a res-representation then it has a syl-representation.

Proof. Let \mathcal{R} be an input, a res-representation of a term $t = \alpha^\mu \beta^\nu$. Then $c = \bar{\nu}$ and thus $\bar{c} = \nu$. From Lemma 6, $(\mathcal{R}, F_{\bar{c}})$ is a (sorted-flushed) syl-representation of the term t . \square

6.4 If a term has a syl-representation then it has a res-representation.

The proof is constructive, that is, it provides an algorithm that takes a syl-representation and produces a res-representation (Algorithm 4). We will again use the key lemma (Lemma 6 from the previous subsection) that establishes a crucial relationship between the two representations (syl and res). In order to use the key lemma, we need to find an algorithm that transforms a given syl-representation into a sorted-flushed syl-representation. We will describe such an algorithm in this subsection. We will divide, naturally, the algorithm into two subalgorithms.

- Algorithm 2 (*Sort*):

It transforms a syl-representation of a term into a sorted syl-representation of the term. It essentially carries out bubble sort.

- Algorithm 3 (*Flush*):

It transforms a sorted syl-representation of a term into a sorted-flushed syl-representation of the term. It essentially carries out repeated swapping of entries of the syl-representation to make it flushed while remaining sorted syl-representation.

Most of this subsection will be devoted in describing and proving the correctness of the two subalgorithms. Now we plunge into details.

Algorithm 2 (*Sort*).

In: $(\mathcal{S}_1, \mathcal{S}_2)$, a syl-representation of a term t

Out: $(\mathcal{S}'_1, \mathcal{S}'_2)$, a sorted syl-representation of the term t

1. $(\mathcal{S}'_1, \mathcal{S}'_2) \leftarrow (\mathcal{S}_1, \mathcal{S}_2)$
2. Repeat
 - (a) $C \leftarrow \text{acs}(\mathcal{S}'_1)$
 - (b) If C is in increasing order then exit the Repeat loop
 - (c) Find $j \in \{1, \dots, n-1\}$ such that $C_j > C_{j+1}$
 - (d) $h \leftarrow C_j - C_{j+1}$
 - (e) Repeat h times
 - i. Find $i \in \{1, \dots, m\}$ such that $\mathcal{S}'_{1,i,j} = 1$ and $\mathcal{S}'_{1,i,j+1} = 0$
 - ii. Swap $\mathcal{S}'_{1,i,j}$ and $\mathcal{S}'_{1,i,j+1}$
3. Repeat
 - (a) $R \leftarrow \text{ars}(\mathcal{S}'_2)$
 - (b) If R is in increasing order then exit the Repeat loop
 - (c) Find $i \in \{1, \dots, m-1\}$ such that $R_i > R_{i+1}$
 - (d) $h \leftarrow R_i - R_{i+1}$
 - (e) Repeat h times
 - i. Find $j \in \{1, \dots, n\}$ such that $\mathcal{S}'_{2,i,j} = 1$ and $\mathcal{S}'_{2,i+1,j} = 0$
 - ii. Swap $\mathcal{S}'_{2,i,j}$ and $\mathcal{S}'_{2,i+1,j}$
4. Return $(\mathcal{S}'_1, \mathcal{S}'_2)$

Example 19. We trace the algorithm *Sort* on the following input.

$$\mathbf{In:} \quad (\mathcal{S}_1, \mathcal{S}_2) = \left[\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array} \right], \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array} \right],$$

which is a syl-representation of the term $t = \alpha_1^3 \alpha_2^3 \alpha_3^3 \alpha_4^2 \alpha_5^2 \beta_1^2 \beta_2^2 \beta_3^2 \beta_4^1$.

1. $(\mathcal{S}'_1, \mathcal{S}'_2) = (\mathcal{S}_1, \mathcal{S}_2)$
2. Iteration 1
 - (a) $C = [2, 6, 8, 7]$
 - (b) C is not sorted
 - (c) $j = 3$
 - (d) $h = 1$
 - i. $i = 3$

ii. Swap $S'_{1,3,3}$ and $S'_{1,3,4}$

$$S'_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Iteration 2

(a) $C = [2, 6, 7, 8]$

(b) C is sorted in increasing order

3. *Iteration 1*

(a) $R = [1, 3, 5, 4, 9]$

(b) R is not sorted

(c) $i = 3$

(d) $h = 1$

i. $j = 1$

ii. Swap $S'_{2,3,1}$ and $S'_{2,4,1}$

$$S'_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Iteration 2

(a) $R = [1, 3, 4, 5, 9]$

(b) R is sorted in increasing order

4. Return (S'_1, S'_2)

$$\text{Out: } (S'_1, S'_2) = \left[\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \right],$$

which is a sorted syl-representation of the term t .

Lemma 8. *The algorithm 2 (Sort) is correct.*

Proof. Let (S_1, S_2) be an input, that is, a syl-representation of a term $t = \alpha^\mu \beta^\nu$. The correctness of the algorithm is immediate from the following claims.

C1: *Right after Step 2, (S'_1, S'_2) is a syl-representation of the term t and $\text{acs}(S'_1)$ is sorted in increasing order.* The proof of the claim is immediate from the following sub-claims.

1. *Right before Step 2a, (S'_1, S'_2) is a syl-representation of the term t .*

We prove it by induction on the number of iterations. At the first iteration, it is trivially true since $(S'_1, S'_2) = (S_1, S_2)$. We assume that it is true after some number of iterations. We need to show that it is still true after one more iteration.

It is immediate from the following observations.

- $rs(S'_1) = \mu$.
Obvious since the loop body does not change $rs(S'_1)$.
- $cs(S'_2) = \nu$.
Obvious since the loop body does not change S'_2 .
- $PC(S'_1, S'_2)$.
Let $A, B \in \{1, \dots, m\}$ such that $A = C_j$ and $B = C_{j+1}$. From Step 2d we know that $C_j = C_{j+1} + h$. Inside Step 2e, we have increased C_{j+1} by h and decreased C_j by h . Thus after Step 2e we have

$$\begin{aligned} C_{j+1} &\leftarrow C_{j+1} + h \\ C_j &\leftarrow C_j - h \end{aligned}$$

Hence $C_j = B$ and $C_{j+1} = A$, that is, we have swapped C_j and C_{j+1} . In other words, the loop body does not change C as a set. Thus $PC(S'_1, S'_2)$ still holds.

2. *The repeat loop in Step 2 terminates.*

It is a bubble sort algorithm executed on a finite list C . Therefore it terminates.

3. *In Step 2c, there exists $j \in \{1, \dots, n-1\}$ such that $C_j > C_{j+1}$.*

The claim is immediate from the following observations.

- Since we are at Step 2c, the ‘if’ condition in Step 2b is not satisfied. Hence C is not in increasing order. Thus there exists $j \in \{1, \dots, n-1\}$ such that $C_j \geq C_{j+1}$.
 - Since (S'_1, S'_2) is a syl-representation, we have $C_j \neq C_{j+1}$.
4. *In Step 2e(i), there exists $i \in \{1, \dots, m\}$ such that $S'_{1,i,j} = 1$ and $S'_{1,i,j+1} = 0$.*
From Step 2d we know $h = C_j - C_{j+1}$. Hence there must exist h different $i \in \{1, \dots, m\}$ such that $S'_{1,i,j} = 1$ and $S'_{1,i,j+1} = 0$.

C2: *Right after Step 3, (S'_1, S'_2) is a syl-representation of the term t and $acs((S'_1))$ and $ars(S'_2)$ are sorted in increasing order.* The proof of the claim is symmetric to the proof of C1. One only needs to switch the roles of S'_1 and S'_2 and the roles of columns and rows.

□

Algorithm 3 (*Flush*).

In: (S_1, S_2) , a sorted syl-representation of a term t

Out: (S'_1, S'_2) , a sorted-flushed syl-representation of the term t

1. $(S'_1, S'_2) \leftarrow (S_1, S_2)$
2. Repeat
 - (a) If (S'_1, S'_2) is flushed then return (S'_1, S'_2)
 - (b) $c \leftarrow cs(S'_1)$
 $r \leftarrow rs(S'_2)$
 $C \leftarrow acs(S'_1)$
 $R \leftarrow ars(S'_2)$

- (c) Find $(i, j) \in \{1, \dots, m-1\} \times \{1, \dots, n\}$ such that $\mathcal{S}'_{2,i,j} = 1$ and $\mathcal{S}'_{2,i+1,j} = 0$
 Swap $\mathcal{S}'_{2,i,j}$ and $\mathcal{S}'_{2,i+1,j}$
- (d) $i_\ell \leftarrow \min \{k \mid r_k = r_i, k \leq i\}$
- (e) If $i_\ell < i$ then
 Find $j \in \{1, \dots, n\}$ such that $\mathcal{S}'_{2,i_\ell,j} = 1$ and $\mathcal{S}'_{2,i,j} = 0$
 Swap $\mathcal{S}'_{2,i_\ell,j}$ and $\mathcal{S}'_{2,i,j}$
- (f) $i_u \leftarrow \max \{k \mid r_k = r_{i+1}, k \geq i+1\}$
- (g) If $i+1 < i_u$ then
 Find $j \in \{1, \dots, n\}$ such that $\mathcal{S}'_{2,i+1,j} = 1$ and $\mathcal{S}'_{2,i_u,j} = 0$.
 Swap $\mathcal{S}'_{2,i+1,j}$ and $\mathcal{S}'_{2,i_u,j}$
- (h) Find $i \in \{1, \dots, m\}$ and $j_\ell < j_u \in \{1, \dots, n\}$ such that
 $\mathcal{S}'_{1,i,j_\ell} = 0$ and $\mathcal{S}'_{1,i,j_u} = 1$ and $C_{j_\ell} = R_{i_\ell} - 1$ and $C_{j_u} = R_{i_u} + 1$
 Swap $\mathcal{S}'_{1,i,j_\ell}$ and \mathcal{S}'_{1,i,j_u}

Example 20. We trace the algorithm *Flush* on the following input.

$$\text{In: } (\mathcal{S}_1, \mathcal{S}_2) = \left[\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array} \right], \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array} \right],$$

which is a sorted syl-representation of the term $t = \alpha_1^3 \alpha_2^3 \alpha_3^3 \alpha_4^2 \alpha_5^2 \beta_1^2 \beta_2^2 \beta_3^2 \beta_4^1$.

1. $(\mathcal{S}'_1, \mathcal{S}'_2) = (\mathcal{S}_1, \mathcal{S}_2)$

2. Iteration 1

(a) $(\mathcal{S}'_1, \mathcal{S}'_2)$ is not flushed

(b) $c = [1, 4, 4, 4]$

$r = [0, 1, 1, 1, 4]$

$C = [2, 6, 7, 8]$

$R = [1, 3, 4, 5, 9]$

(c) $i = 2, j = 2$

Swap $\mathcal{S}'_{2,2,2}$ and $\mathcal{S}'_{2,3,2}$

$$(\mathcal{S}'_1, \mathcal{S}'_2) = \left[\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array} \right], \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array} \right]$$

(d) $i_\ell = 2$

(e) $i_\ell \not\leq i$

(f) $i_u = 4$

(g) $i+1 < i_u$

$j = 2$

Swap $\mathcal{S}'_{2,3,2}$ and $\mathcal{S}'_{2,4,2}$

$$(\mathcal{S}'_1, \mathcal{S}'_2) = \left[\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array} \right], \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array} \right]$$

(h) $i = 1$ and $j_\ell = 1, j_u = 2$

Swap $\mathcal{S}'_{1,1,1}$ and $\mathcal{S}'_{1,1,2}$

$$(\mathcal{S}'_1, \mathcal{S}'_2) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Iteration 2

(a) $(\mathcal{S}'_1, \mathcal{S}'_2)$ is not flushed

(b) $c = [2, 3, 4, 4]$

$r = [0, 0, 1, 2, 4]$

$C = [3, 5, 7, 8]$

$R = [1, 2, 4, 6, 9]$

(c) $i = 3, j = 3$

Swap $\mathcal{S}'_{2,3,3}$ and $\mathcal{S}'_{2,4,3}$

$$(\mathcal{S}'_1, \mathcal{S}'_2) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(d) $i_\ell = 3$

(e) $i_\ell \not\prec i$

(f) $i_u = 4$

(g) $i + 1 \not\prec i_u$

(h) $i = 2$ and $j_\ell = 1, j_u = 3$

Swap $\mathcal{S}'_{1,2,1}$ and $\mathcal{S}'_{1,2,3}$

$$(\mathcal{S}'_1, \mathcal{S}'_2) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Iteration 3

(a) $(\mathcal{S}'_1, \mathcal{S}'_2)$ is flushed

$$\mathbf{Out:} (\mathcal{S}'_1, \mathcal{S}'_2) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

which is a sorted-flushed syl-representation of the term t .

Lemma 9. The algorithm 3 (Flush) is correct.

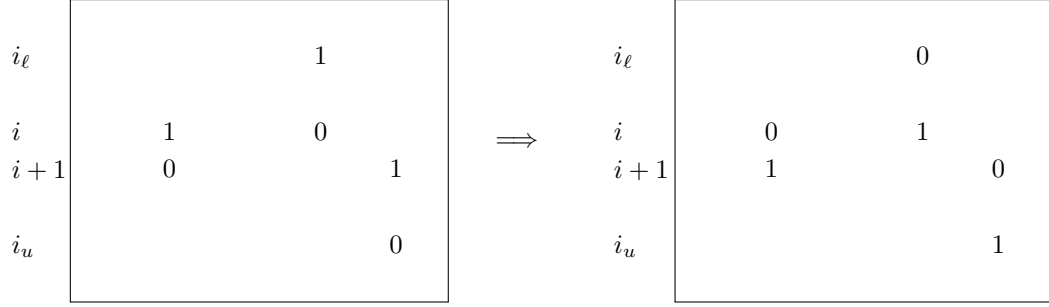
Proof. Let $(\mathcal{S}_1, \mathcal{S}_2)$ be an input, that is, a sorted syl-representation of a term $t = \alpha^\mu \beta^\nu$. The correctness of the algorithm is immediate from the following claims.

C1: Right before Step 2a, $(\mathcal{S}'_1, \mathcal{S}'_2)$ is a sorted syl-representation of the term t .

We prove it by induction on the number of iterations. At the first iteration, it is trivially true since $(\mathcal{S}'_1, \mathcal{S}'_2) = (\mathcal{S}_1, \mathcal{S}_2)$. We assume that it is true after some number of iterations. We need to show that it is still true after one more iteration. It is immediate from the following two sub-claims, where $C' = \text{acs}(\mathcal{S}'_1)$ and $R' = \text{ars}(\mathcal{S}'_2)$ at the end of Step 2h.

- C' and R' are sorted in increasing order.

Note that Steps 2c–2g transform \mathcal{S}'_2 by carrying out swaps along columns, as depicted by the following diagram.



Thus

$$R'_t = \begin{cases} R_t - 1 & \text{if } t = i_\ell \\ R_t + 1 & \text{if } t = i_u \\ R_t & \text{else} \end{cases}$$

From Step 2h, it is immediate that

$$C'_t = \begin{cases} C_t + 1 & \text{if } t = j_\ell \\ C_t - 1 & \text{if } t = j_u \\ C_t & \text{else} \end{cases}$$

Next recall that R and C are sorted in increasing order. Thus we only need to show that $R'_{i_\ell} > R_{i_\ell-1}$, $R'_{i_u} < R_{i_u+1}$, $C'_{j_\ell} < C_{j_\ell+1}$ and $C'_{j_u} > C_{j_u-1}$. We show them one by one.

- $R'_{i_\ell} > R_{i_\ell-1}$
Recall that $R'_{i_\ell} = R_{i_\ell} - 1$. Since $r_{i_\ell} > r_{i_\ell-1}$ we have $R_{i_\ell} - 1 > R_{i_\ell-1}$. Thus $R'_{i_\ell} > R_{i_\ell-1}$.
 - $R'_{i_u} < R_{i_u+1}$
Recall that $R'_{i_u} = R_{i_u} + 1$. Since $r_{i_u} < r_{i_u+1}$ we have $R_{i_u} + 1 < R_{i_u+1}$. Thus $R'_{i_u} < R_{i_u+1}$.
 - $C'_{j_\ell} < C_{j_\ell+1}$
Recall that $C'_{j_\ell} = C_{j_\ell} + 1$. Since $C_{j_\ell} = R_{i_\ell} - 1$ we have $C'_{j_\ell} = R_{i_\ell}$. Since R_{i_ℓ} appears in R and $PC(C, R)$, we see that R_{i_ℓ} does not appear in C . Hence $R_{i_\ell} < C_{j_\ell+1}$. Thus $C'_{j_\ell} < C_{j_\ell+1}$.
 - $C'_{j_u} > C_{j_u-1}$
Recall that $C'_{j_u} = C_{j_u} - 1$. Since $C_{j_u} = R_{i_u} + 1$ we have $C'_{j_u} = R_{i_u}$. Since R_{i_u} appears in R and $PC(C, R)$, we see that R_{i_u} does not appear in C . Hence $R_{i_u} > C_{j_u-1}$. Thus $C'_{j_u} > C_{j_u-1}$.
- $(\mathcal{S}'_1, \mathcal{S}'_2)$ is a syl-representation of the term t at the end of Step 2h.
 - $rs(\mathcal{S}'_1) = \mu$.
Obvious since the loop body does not change $rs(\mathcal{S}'_1)$.
 - $cs(\mathcal{S}'_2) = \nu$.
Obvious since the loop body does not change $cs(\mathcal{S}'_2)$.
 - $PC(\mathcal{S}'_1, \mathcal{S}'_2)$ holds.
Note $R'_{i_\ell} = C_{j_\ell}$, $R'_{i_u} = C_{j_u}$ and $C'_{j_\ell} = R_{i_\ell}$, $C'_{j_u} = R_{i_u}$. Note that the others do not change. In other words $C' \cup R' = C \cup R$ as sets. Thus $PC(\mathcal{S}'_1, \mathcal{S}'_2)$ holds.

C2: *The main loop (Repeat) terminates.*

For every iteration of the main loop, at least one “out of order” pair of $(1, 0)$ in \mathcal{S}_2 is swapped. Thus the algorithm terminates.

C3: *In Step 2c, there exists (i, j) satisfying the conditions stated in the step.*

Since we are at Step 2c, the ‘if’ condition in Step 2a is not satisfied. Hence $(\mathcal{S}'_1, \mathcal{S}'_2)$ is not flushed. Thus there exists $(i, j) \in \{1, \dots, m-1\} \times \{1, \dots, n\}$ such that $\mathcal{S}'_{2,i,j} = 1$ and $\mathcal{S}'_{2,i+1,j} = 0$.

C4: *In Step 2e, there exists j satisfying the conditions stated in the step.*

Let $r' = rs(\mathcal{S}'_2)$ right before entering the step. Note

- From Step 2c, we have $r'_i = r_i - 1$.
- From Step 2d, we have $r'_{i_\ell} = r_{i_\ell} = r_i$.

Therefore there exist $j \in \{1, \dots, n\}$ such that $\mathcal{S}'_{2,i,j} = 0$ and $\mathcal{S}'_{2,i_\ell,j} = 1$.

C5: *In Step 2g, there exists j satisfying the conditions stated in the step.*

Let $r' = rs(\mathcal{S}'_2)$ right before entering the step. Note

- From Step 2c, we have $r'_{i+1} = r_{i+1} + 1$.
- From Step 2f, we have $r'_{i_u} = r_{i_u} = r_{i+1}$.

Therefore there exist $j \in \{1, \dots, n\}$ such that $\mathcal{S}'_{2,i+1,j} = 1$ and $\mathcal{S}'_{2,i_u,j} = 0$.

C6: *In Step 2h, there exist i, j_ℓ, j_u satisfying the conditions stated in the step.*

1. From Step 2d, we have

- if $i_\ell = 1$ then $R_{i_\ell} \geq 2$ (since $\mathcal{S}'_{2,i,j} = 1$ and in turn $r_{i_\ell} = r_i \geq 1$).
- if $i_\ell > 1$ then $R_{i_\ell} - R_{i_\ell-1} \geq 2$ (since $r_{i_\ell} \geq r_{i_\ell-1} + 1$).

Thus $R_{i_\ell} - 1$ does not appear in R . Hence it must appear in C . Thus there exists j_ℓ such that $C_{j_\ell} = R_{i_\ell} - 1$.

2. From Step 2f, we have

- if $i_u = m$ then $R_{i_u} \leq m + n - 1$ (since $\mathcal{S}'_{2,i+1,j} = 0$ and in turn $r_{i_u} = r_{i+1} \leq n - 1$).
- if $i_u < m$ then $R_{i_u+1} - R_{i_u} \geq 2$ (since $r_{i_u} + 1 \leq r_{i_u+1}$).

Thus $R_{i_u} + 1$ does not appear in R . Hence it must appear in C . Thus there exists j_u such that $C_{j_u} = R_{i_u} + 1$.

Therefore there exist $j_\ell, j_u \in \{1, \dots, n\}$ such that $C_{j_\ell} = R_{i_\ell} - 1$ and $C_{j_u} = R_{i_u} + 1$.

It remains to show that there exists i that satisfies the conditions of Step 2h.

Note that $R_{i_\ell}, R_{i_\ell+1}, \dots, R_{i_u}$ appear in R . Hence they do not appear in C . Note that

$$\begin{aligned} R_{i_\ell} &= r_i + i_\ell \\ R_{i_\ell+1} &= r_i + i_\ell + 1 \\ &\vdots \\ R_i &= r_i + i \\ R_{i+1} &= r_{i+1} + i + 1 \\ R_{i+2} &= r_{i+1} + i + 2 \end{aligned}$$

$$\begin{aligned} & \vdots \\ R_{i_u} &= r_{i+1} + i_u \end{aligned}$$

Note that R_{i_ℓ}, \dots, R_i are consecutive integers. Likewise note that R_{i+1}, \dots, R_{i_u} are consecutive integers. We show that $j_u - j_\ell = R_{i+1} - R_i$. Consider the following two cases:

Case 1: $R_i + 1 = R_{i+1}$

Note that $R_{i_\ell}, \dots, R_{i_u}$ are consecutive and they do not appear in C . Since C is sorted in increasing order and $C_{j_\ell} = R_{i_\ell} - 1$ and $C_{j_u} = R_{i_u} + 1$, there is nothing in between C_{j_ℓ} and C_{j_u} . Hence $j_u - j_\ell = 1$. Note that $R_{i+1} - R_i = 1$. Thus $j_u - j_\ell = R_{i+1} - R_i$.

Case 2: $R_i + 1 < R_{i+1}$

Note that the consecutive list of numbers $R_i + 1, \dots, R_{i+1} - 1$ do not appear in R . Hence they appear in C . Since C is sorted in increasing order and $C_{j_\ell} = R_{i_\ell} - 1$ and $C_{j_u} = R_{i_u} + 1$, we conclude that exactly $R_i + 1, \dots, R_{i+1} - 1$ appear in between C_{j_ℓ} and C_{j_u} . Hence

$$j_u - j_\ell - 1 = (R_{i+1} - 1) - (R_i + 1) + 1 = R_{i+1} - R_i - 1$$

Thus $j_u - j_\ell = R_{i+1} - R_i$.

In both cases, we have shown that $j_u - j_\ell = R_{i+1} - R_i$. Note

$$\begin{aligned} c_{j_u} - c_{j_\ell} &= (C_{j_u} - j_u) - (C_{j_\ell} - j_\ell) \\ &= (C_{j_u} - C_{j_\ell}) - (j_u - j_\ell) \\ &= ((R_{i_u} + 1) - (R_{i_\ell} - 1)) - (R_{i+1} - R_i) \\ &= ((r_{i+1} + i_u + 1) - (r_i + i_\ell - 1)) - ((r_{i+1} + i + 1) - (r_i + i)) \\ &= i_u - i_\ell + 1 \\ &\geq 2 \end{aligned}$$

Hence there exists $i \in \{1, \dots, m\}$ such that $\mathcal{S}'_{1,i,j_\ell} = 0$ and $\mathcal{S}'_{1,i,j_u} = 1$.

□

Algorithm 4 (*ResFromSyl*).

In: $(\mathcal{S}_1, \mathcal{S}_2)$, a *syl*-representation of a term t

Out: \mathcal{R} , a *res*-representation of the term t

1. $(\mathcal{S}'_1, \mathcal{S}'_2) \leftarrow \text{Sort}(\mathcal{S}_1, \mathcal{S}_2)$
2. $(\mathcal{S}'_1, \mathcal{S}'_2) \leftarrow \text{Flush}(\mathcal{S}'_1, \mathcal{S}'_2)$
3. $\mathcal{R} \leftarrow \mathcal{S}'_1$
4. *Return* \mathcal{R} .

Example 21. We trace the algorithm *ResFromSyl* on the following input.

$$\text{In: } (\mathcal{S}_1, \mathcal{S}_2) = \left[\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array} \right], \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array} \right],$$

which is a *syl*-representation of the term $t = \alpha_1^3 \alpha_2^3 \alpha_3^3 \alpha_4^2 \alpha_5^2 \beta_1^2 \beta_2^2 \beta_3^2 \beta_4^1$.

$$\begin{aligned}
1. (\mathcal{S}'_1, \mathcal{S}'_2) &= \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \\
2. (\mathcal{S}'_1, \mathcal{S}'_2) &= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \\
3. \mathcal{R} &= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \\
\text{Out: } \mathcal{R} &= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix},
\end{aligned}$$

which is a res-representation of the term t .

Lemma 10. *The algorithm 4 (ResFromSyl) is correct. Thus if a term has a syl-representation then it has a res-representation.*

Proof. Let $(\mathcal{S}_1, \mathcal{S}_2)$ be an input, that is, a syl-representation of a term t . The correctness of the algorithm is immediate from the following claims.

C1: *After Step 1, $(\mathcal{S}'_1, \mathcal{S}'_2)$ is a sorted syl representation of the term t .*

Immediate from the specification of Algorithm 2 (*Sort*) and Lemma 8.

C2: *After Step 2, $(\mathcal{S}'_1, \mathcal{S}'_2)$ is a sorted-flushed syl-representation of the term t .*

Immediate from C1, the specification of Algorithm 3 (*Flush*) and Lemma 9.

C3: *After Step 3, \mathcal{R} is a res representation of the term t*

Immediate from C2 and Lemma 6.

□

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